PhD Thesis

Topics in harmonic analysis related to weighted Poincaré–Sobolev inequalities

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics and Statistics at the University of the Basque Country

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A Carmen.

Abstract

This thesis develops several topics in harmonic analysis, with weighted Poincaré–Sobolev inequalities as its main theme. The first part of the thesis studies generalized Poincaré inequalities formulated through local oscillation estimates controlled by functionals on cubes. We use a geometric summability condition, called the $SD_p^s(w)$ condition, to derive generalized Poincaré–Sobolev type estimates. We then study several applications and extensions of this abstract result.

Another central part focuses on sharp quantitative weighted Poincaré–Sobolev inequalities obtained via representation formulas and weighted estimates for Riesz potentials. This approach yields sharp dependence on Muckenhoupt constants and allows us to identify the optimal weighted Sobolev exponent associated with a fixed weight. We also investigate quantitative embeddings of A_∞ into A_p for flat weights and derive corresponding Poincaré–Sobolev inequalities for A_∞ weights with small Fujii–Wilson constant.

In addition to these main topics, the thesis contains some related contributions. We prove quantitative BMO–BLO estimates for the Hardy–Littlewood maximal operator, including weighted exponential estimates and a characterization of A_∞ through the mapping $\mathcal{M} : \mathrm{BMO} \to \mathrm{BLO}_w$. We establish a pointwise Bourgain–Brezis–Mironescu formula for Riesz potentials involving a nonlinear fractional differential operator. Finally, we study harmonic analysis with respect to Hausdorff content, proving capacitary Fefferman–Stein and Muckenhoupt–Wheeden inequalities and characterizing BMO^β through commutators of fractional integrals.

Resumen

Esta tesis desarrolla varios temas dentro del análisis armónico, teniendo como tema principal las desigualdades de Poincaré–Sobolev con pesos. En la primera parte de la tesis estudiamos desigualdades de Poincaré generalizadas formuladas mediante estimaciones de oscilación local controladas por un funcional definido sobre cubos. Utilizamos una condición geométrica de sumabilidad, llamada condición $SD_p^s(w)$, para obtener desigualdades generalizadas de tipo Poincaré–Sobolev. Después estudiamos varias aplicaciones y extensiones de este resultado abstracto.

Otra parte central de la tesis trata sobre desigualdades de Poincaré–Sobolev con pesos y dependencia cuantitativa óptima, obtenidas mediante fórmulas de representación y estimaciones con pesos para el potencial de Riesz. Este enfoque proporciona dependencia óptima respecto de las constantes de Muckenhoupt y nos permite identificar el exponente de Sobolev óptimo asociado a un peso fijo. También estudiamos la inclusión cuantitativa de A_∞ en A_p para pesos planos y obtenemos las correspondientes desigualdades de Poincaré–Sobolev para pesos A_∞ con constante de Fujii–Wilson pequeña.

Además de estos temas principales, la tesis contiene algunas contribuciones relacionadas. Probamos estimaciones cuantitativas BMO–BLO para el operador maximal de Hardy–Littlewood, incluyendo estimaciones exponenciales con pesos y una caracterización de A_∞ mediante la acotación $\mathcal{M} : \mathrm{BMO} \to \mathrm{BLO}_w$. Establecemos una fórmula puntual de Bourgain–Brezis–Mironescu para potenciales de Riesz que involucra un operador diferencial fraccionario no lineal. Finalmente, estudiamos análisis armónico respecto del contenido de Hausdorff, probando desigualdades capacitarias de Fefferman–Stein y Muckenhoupt–Wheeden, y caracterizando BMO^β mediante conmutadores de integrales fraccionarias.

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Notation

Symbol	Meaning
Sets and geometry	
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{Z}^n$	the n-fold product of $\mathbb{Z}$
$\mathbb{R}^n$	the n-dimensional Euclidean space
$\lvert x\rvert$	$\sqrt{x_1^2 + \cdots + x_n^2}$ when $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
$\operatorname{dist}(x, A)$	the distance from the point x to the set A
supp	the support
$B(x, r)$	the Euclidean ball of center x with radius r
Q	a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes
$Q(x, r)$	the cube centered at x with side length r and sides parallel to the coordinate axes
$\mathcal{Q}$	the set of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes
$\ell(Q)$	the side length of the cube Q
Measure and basic functions	
dx, dy, dz, dt	integration with respect to Lebesgue measure
$\lvert A\rvert$	the Lebesgue measure of a set $A \subset \mathbb{R}^n$
μ	a measure
w	a weight
$\mathcal{H}^\beta$	the β-dimensional Hausdorff measure

Symbol	Meaning
$\mathcal{H}_\infty^\beta$	the β-dimensional Hausdorff content
$\mathcal{H}_\infty^{\beta,Q_0}$	the dyadic β-dimensional Hausdorff content associated with the cube Q_0
$\mathrm{cap}_{\alpha,p}$	the Riesz (α, p)-capacity
χ_A	the characteristic function of a set $A \subset \mathbb{R}^n$
f_Q	the integral average of f over Q, namely $\frac{1}{\lvert Q\rvert}\int_Q f\,dx$
$f_{Q,\mu}$	the integral average of f over Q with respect to the measure μ, namely $\frac{1}{\mu(Q)}\int_Q f\,d\mu$
inf	the infimum
sup	the supremum
ess inf	the essential infimum
ess sup	the essential supremum
log	the natural logarithm
$\log^+$	the function $\max\{\log, 0\}$
f^+	the positive part of the function f, namely $\max\{f, 0\}$
f^-	the negative part of the function f, namely $\max\{-f, 0\}$
$\lfloor x \rfloor$	the integer part of x, namely the greatest integer less than or equal to x

Function spaces

$\mathrm{Lip}(\mathbb{R}^n)$	the space of Lipschitz functions
$C^k(\mathbb{R}^n)$	the space of k-times continuously differentiable functions
$C^\infty(\mathbb{R}^n)$	the space of smooth functions
$C_c^\infty(\mathbb{R}^n)$	the space of smooth functions with compact support
$\mathcal{S}(\mathbb{R}^n)$	the Schwartz class
$W^{1,p}(\mathbb{R}^n)$	the Sobolev space
$W^{\delta,p}(\mathbb{R}^n)$	the fractional Sobolev space
$L^p(X,\mu)$	the Lebesgue space over the measure space (X, μ)
$L^{p,q}(X,\mu)$	the Lorentz space over the measure space (X, μ)
$L^{p,\infty}(X,\mu)$	the weak-Lebesgue space over the measure space (X, μ)
$L^1_{\mathrm{loc}}(\mathbb{R}^n)$	the space of locally integrable functions in $\mathbb{R}^n$
BMO	the space of functions of bounded mean oscillation
BLO	the space of functions of bounded lower oscillation
BMO_w	the weighted BMO space
BLO_w	the weighted BLO space
BMO^β	the space of functions of bounded β-dimensional mean oscillation
$\mathscr{M}^\beta(\mathbb{R}^n)$	the Morrey space

Symbol	Meaning
$\|f\|_{L^p(\mu)}$	the norm of f in the space $L^p(X,\mu)$
$\|f\|_{L^{p,q}(\mu)}$	the norm of f in the Lorentz space $L^{p,q}(X,\mu)$
$\|f\|_{L^{p,\infty}(\mu)}$	the quasi-norm of f in the space $L^{p,\infty}(X,\mu)$
$\|f\|_{\mathrm{BMO}}$	the seminorm of f in the BMO space
$\|f\|_{W^{1,p}}$	the norm of f in the Sobolev space $W^{1,p}(\mathbb{R}^n)$
$\|f\|_{W^{\delta,p}}$	the norm of f in the fractional Sobolev space $W^{\delta,p}(\mathbb{R}^n)$
$\|\mu\|_{\mathscr{M}^\beta(\mathbb{R}^n)}$	the Morrey norm of the measure μ
$P_Q^m f$	a polynomial approximation of f on Q of degree at most m
p'	the conjugate exponent of $p \in [1,\infty]$

Differential operators

∂_j	the partial derivative with respect to the j-th variable
∇	the gradient
div	the divergence
Δ	the Laplacian
$\mathcal{D}^\alpha$	the nonlinear fractional differential operator of order α
$\mathcal{D}_Q^\alpha$	the local nonlinear fractional differential operator of order α associated with the cube Q
$\mathcal{D}_p^\alpha$	the L^p nonlinear fractional differential operator of order α

Maximal and integral operators

$\mathcal{M}$	the Hardy–Littlewood maximal operator
$\mathcal{M}^c$	the centered Hardy–Littlewood maximal operator
$\mathcal{M}_d$	the dyadic Hardy–Littlewood maximal operator
$\mathcal{M}_{Q_0}$	the local dyadic Hardy–Littlewood maximal operator with respect to Q_0
$\mathcal{M}_\alpha$	the fractional maximal operator of order α
$\mathcal{M}^\#$	the Fefferman–Stein sharp maximal operator
$\mathcal{M}_{Q_0}^\#$	the local dyadic Fefferman–Stein sharp maximal operator
I_α	the Riesz potential of order α
$[b,T]$	the commutator, defined by $[b,T](\cdot) = bT(\cdot) - T(b\cdot)$
$\widehat{f}$	the Fourier transform of f

Weights

A_p	the Muckenhoupt class of weights
A_∞	the Muckenhoupt class $\bigcup_{p\geq 1} A_p$
$A_{p,\mathcal{R}}$	the strong Muckenhoupt class of weights associated with rectangles

Symbol	Meaning
RH_q	the Reverse Hölder class of order q
$[w]_{A_p}$	the A_p constant of the weight w
$[w]_{A_{p,q}}$	the characteristic constant of the Chung–Hunt–Kurtz class of weights $A_{p,q}$
$[w]_{A_\infty}$	the Fujii–Wilson A_∞ constant of the weight w
$[w]'_{A_\infty}$	the Hruščev A_∞ constant of the weight w
$[w]^{\log}_{A_\infty}$	the logarithmic A_∞ constant of the weight w
$[w]_{A_{p,\mathcal{R}}}$	the $A_{p,\mathcal{R}}$ constant of the weight w

Cube functionals and summability conditions

a	a functional defined on cubes
$D_p(w)$	the weighted D_p condition for cube functionals
$SD_p^s(w)$	the weighted $SD_p^s(w)$ condition for cube functionals
$\|a\|_{D_p(w)}$	the best constant in the $D_p(w)$ condition
$\|a\|_{SD_p^s(w)}$	the best constant in the $SD_p^s(w)$ condition

Capacitary operators

$\mathcal{M}_{\mathcal{H}_\infty^\beta}$	the maximal operator associated with $\mathcal{H}_\infty^\beta$
$\mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0}}$	the dyadic maximal operator associated with $\mathcal{H}_\infty^{\beta,Q_0}$
$\mathcal{M}_\beta^\#$	the β-dimensional sharp maximal operator
$\mathcal{M}_{\beta,Q_0}^\#$	the dyadic β-dimensional sharp maximal operator associated with the cube Q_0
$\mathcal{M}_{\alpha,\mathcal{H}_\infty^\beta}$	the fractional maximal operator of order α associated with $\mathcal{H}_\infty^\beta$
$\mathcal{M}_{\alpha,B,\mathcal{H}_\infty^\beta}$	the Orlicz fractional maximal operator of order α associated with $\mathcal{H}_\infty^\beta$

Equivalence relations and bounds

$A \lesssim B$	$A \le CB$ for some positive constant C
$A \lesssim_\alpha B$	$A \le CB$ for some positive constant C, depending only on α
$A \simeq B$	$A \lesssim B$ and $B \lesssim A$
$A \simeq_\alpha B$	$A \lesssim_\alpha B$ and $B \lesssim_\alpha A$
C (or c)	positive constants whose exact values may change from line to line
$C(\alpha,\beta,\dots)$ or $C_{\alpha,\beta,\dots}$	constants depending only on $\alpha,\beta,\dots$, whose values may change from line to line

Introduction

In this thesis, I present the research that I have carried out during my four years as a PhD student, from September 2022 to July 2026, under the supervision of Carlos Pérez. The main topic of this thesis is the study of weighted Poincaré–Sobolev inequalities, with particular emphasis on quantitative estimates, self-improving phenomena, and Muckenhoupt weights. However, during these years I have also worked on some related, and sometimes less related, problems in harmonic analysis. This thesis aims to be reasonably self-contained: Chapter 1 collects the notation, definitions, and preliminary results that will be used repeatedly throughout the thesis, while each of the subsequent chapters contains its own introduction and motivation. In the remainder of this introduction, we briefly describe the main topics of the thesis chapter by chapter.

Chapter 2 is devoted to self-improving properties of generalized Poincaré inequalities. This line of research goes back to the seminal work of Franchi, Pérez, and Wheeden **[FPW98]**. The starting point of our approach is a generalized Poincaré inequality of the form

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le a(Q),$$

for every cube Q, where $f_Q = \frac{1}{|Q|}\int_Q f(y)\,dy$ and a is a functional defined on cubes. This formulation models several different types of inequalities. For instance, if $f \in \mathrm{BMO}$, then we may simply take $a(Q) = \|f\|_{\mathrm{BMO}}$ for every cube Q. It also includes the classical local $(1,p)$ Poincaré inequality,

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le c_n\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Similarly, higher-order Poincaré inequalities are also covered by this framework, after replacing the average f_Q with an appropriate polynomial approximation $P_Q^m f$,

$$\frac{1}{|Q|}\int_Q |f(x)-P_Q^m f(x)|\,dx \le c_{n,m}\,\ell(Q)^m\left(\frac{1}{|Q|}\int_Q |\nabla^m f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

The same point of view also covers fractional Poincaré inequalities. For $0<\delta<1$,

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le c_n(1-\delta)^{\frac{1}{p}}\,\ell(Q)^{\delta}\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}},$$

where the factor $(1-\delta)^{\frac{1}{p}}$ is the extra gain coming from the Bourgain–Brezis–Mironescu phenomenon **[BBM01]**.

The common feature behind these examples is that the corresponding functional satisfies a geometric summability condition. As we shall see in this chapter, this condition is precisely what implies the self-improving phenomenon. More concretely, we will use the $SD_p^s(w)$ condition, introduced by Pérez and Rela **[PR19]**. Given a weight w in $\mathbb{R}^n$, $s > 0$, and $0 < p < \infty$, we say that the functional a satisfies the $SD_p^s(w)$ condition if there exists a constant $C > 0$ such that, for every cube Q and every family $\{Q_j\}_j$ of pairwise disjoint subcubes of Q, one has

$$\sum_j a(Q_j)^p w(Q_j) \leq C^p \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|} \right)^{\frac{p}{s}} a(Q)^p w(Q).$$

Under this condition, and assuming that $w \in A_r$, we prove that the initial estimate self-improves in the subcritical range $p < rs$. More precisely, if $p^*_{r,s}$ is defined by

$$\frac{1}{p} - \frac{1}{p^*_{r,s}} = \frac{1}{rs},$$

then we obtain the following generalized Poincaré–Sobolev inequality

$$\left\| f - f_Q \right\|_{L^{p^*_{r,s},\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \leq c_n \, p^*_{r,s} \, [w]_{A_\infty} \, [w]_{A_r}^{\frac{1}{rs}} \, \|a\|_{SD_p^s(w)} \, a(Q),$$

for every cube Q, where $\|a\|_{SD_p^s(w)}$ is the best constant in the $SD_p^s(w)$ condition. We also treat the endpoint and supercritical regimes $p \geq rs$, where the corresponding self-improvement gives exponential integrability instead of an $L^{p^*_{r,s},\infty}$ estimate.

The second part of Chapter 2 contains applications and extensions of this abstract result. We first present three extensions of the general self-improving theorem: to oscillation estimates involving polynomial projections, to vector-valued functions, and to the setting of rectangles. The remaining sections focus on applications to weighted Poincaré–Sobolev inequalities, fractional Poincaré–Sobolev inequalities, including two-weight versions, endpoint exponential estimates, and Poincaré–Sobolev inequalities involving Lorentz norms of the gradient. Finally, by means of a completely different method, we study Poincaré–Sobolev inequalities in which weak norms appear on both sides of the estimate.

In Chapter 3, we study weighted Poincaré–Sobolev inequalities via fractional integration. This chapter follows a different approach from the self-improving method developed in Chapter 2. The abstract framework studied in Chapter 2 allows one to treat many Poincaré-type inequalities in a unified way, but it does not generally provide sharp dependence on the corresponding weight constants. In Chapter 3, instead of starting from an abstract oscillation inequality and improving it through a discrete summability condition, we use representation formulas and weighted estimates for Riesz potentials. The underlying idea is classical: one controls the oscillation of a Sobolev function by a fractional integral of its gradient, and then applies weighted bounds for the corresponding potential operator. This approach, combined with the truncation method, allows us to obtain weighted Poincaré–Sobolev inequalities with sharp quantitative dependence on the Muckenhoupt constants.

More specifically, let $n \geq 2$, $1 \leq p < n$, and let $w \in A_r$ for some $1 \leq r \leq p$. If $r > 1$, we consider the exponent q_r defined by

$$\frac{1}{p} - \frac{1}{q_r} = \frac{1}{n} \frac{\tau_n [\sigma]_{A_\infty}}{1 + r(\tau_n [\sigma]_{A_\infty} - 1)},$$

where $\sigma = w^{1-r'}$, while in the case $r = 1$ we set $q_1 = p^* = \frac{np}{n-p}$. We prove that there exists $c_n > 0$ such that

$$\left(\frac{1}{w(Q)} \int_Q |f(x) - f_Q|^{q_r} w(x)\, dx \right)^{\frac{1}{q_r}} \leq c_n p^* [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x)\, dx \right)^{\frac{1}{p}},$$

for every cube Q. Moreover, the power $\frac{1}{p}$ on the A_r constant is sharp. In particular, in the case $r = 1$, this gives a positive answer to a conjecture proposed by Pérez and Rela in **[PR19]** concerning the optimal power of the A_1 constant in this inequality.

After establishing the sharp dependence on the A_r constant, we study the optimal exponent that can appear on the left-hand side of the preceding weighted Poincaré–Sobolev inequality for a fixed weight $w \in A_p$. We refer to this quantity as the optimal weighted Sobolev exponent associated with w, and denote it by p_w^*. More precisely, we prove that it is given by

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n} \frac{1}{\ell_w},$$

where

$$\ell_w := \inf\{r \geq 1 : \ w \in A_r\}.$$

Thus, for a given weight $w \in A_\infty$, estimating ℓ_w becomes a natural problem. This motivates the quantitative study of the classical embedding of A_∞ into A_p for some $p \geq 1$, to which the final part of the chapter is devoted. In particular, we prove an asymptotically sharp embedding result for flat weights. As a consequence, we obtain a new Poincaré–Sobolev inequality for A_∞ weights whose Fujii–Wilson constant is small.

Chapter 4 is devoted to quantitative BMO–BLO estimates for the Hardy–Littlewood maximal function. The starting point is the classical result of Bennett, DeVore, and Sharpley **[BDS81]**, which states that the maximal operator $\mathcal{M}$ is bounded on BMO. This was later sharpened by Bennett **[Ben82]**, who proved the stronger mapping property $\mathcal{M} : \mathrm{BMO} \to \mathrm{BLO}$, where BLO is strictly contained in BMO. Our aim in this chapter is to study this boundedness quantitatively and in the weighted setting. One of the main motivations comes from Karagulyan's extension of the John–Nirenberg inequality **[Kar02]**, where the oscillation of an arbitrary nonconstant locally integrable function is normalized by its sharp maximal function. In the same spirit, we obtain exponential estimates for the lower oscillation of $\mathcal{M}f$, normalized by $\mathcal{M}^\# f$. More precisely, if f is a nonconstant locally integrable function such that $\mathcal{M}f$ is not identically infinite and $w \in A_\infty$, then we prove that for every cube Q and every $1 \leq p < \infty$,

$$\left(\frac{1}{w(Q)} \int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f(x)} \right)^p w(x)\, dx \right)^{\frac{1}{p}} \leq c_n\, [w]_{A_\infty}\, p.$$

The linear growth in p on the right-hand side is equivalent to exponential decay for the weighted measure of the corresponding level sets.

In the second part of the chapter, we address the converse question: whether the condition $w \in A_\infty$ is also necessary for this type of estimate. We show that this is indeed the case. More precisely, we prove a new quantitative characterization of the class A_∞ in terms of the mapping property

$$\mathcal{M} : \mathrm{BMO} \longrightarrow \mathrm{BLO}_w.$$

Namely, the above mapping property holds if and only if $w \in A_\infty$. This characterization further shows that the dependence on the weight constant in the previous estimate is optimal: in general, one cannot replace the linear dependence on $[w]_{A_\infty}$ by a better dependence.

In Chapter 5 we prove a pointwise Bourgain–Brezis–Mironescu (BBM) formula for Riesz potentials. Motivated by recent subrepresentation formulas established in **[HMP25a]**, we study the limiting behavior, as $\alpha \to 1^-$, of the operator

$$(1-\alpha) I_\alpha(\mathcal{D}^\alpha f),$$

where $\mathcal{D}^\alpha$ is the nonlinear fractional differential operator introduced by Spector **[Spe20]**, defined by

$$\mathcal{D}^\alpha f(x) = \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+\alpha}}\, dy, \qquad 0 < \alpha < 1.$$

This operator can be viewed as a nonlocal fractional gradient of order α, and the factor $(1-\alpha)$ is the natural BBM normalization **[BBM01]**. The main result of the chapter shows that, for $f \in C_c^\infty(\mathbb{R}^n)$ and $n \geq 2$,

$$\lim_{\alpha \to 1^-} (1-\alpha) I_\alpha(\mathcal{D}^\alpha f)(x) = K_n I_1(|\nabla f|)(x),$$

for every $x \in \mathbb{R}^n$, where K_n is the same geometric constant appearing in the classical BBM formula. We then extend this result to functions in $W^{1,1}(\mathbb{R}^n)$ by a density argument: given any sequence $\alpha_k \to 1^-$, we obtain almost everywhere convergence along a subsequence.

Chapter 6 moves in a direction that is less directly connected to the preceding chapters of the thesis. In this chapter, we study harmonic analysis with respect to Hausdorff content, replacing Lebesgue measure by a capacity, namely the non-additive set function $\mathcal{H}^\beta_\infty$ defined by

$$\mathcal{H}^\beta_\infty(E) := \inf\left\{ \sum_{j=1}^\infty \omega_\beta r_j^\beta : E \subset \bigcup_{j=1}^\infty B(x_j, r_j) \right\},$$

where ω_β is a normalization constant. This point of view is natural in the study of questions involving traces, lower-dimensional phenomena, and endpoint Sobolev embeddings. In this setting, a natural replacement for the classical space BMO is given by the spaces BMO^β, introduced by Chen and Spector **[CS24]**, where the oscillation is measured with respect to $\mathcal{H}^\beta_\infty$. More precisely, for $0 < \beta \leq n$, one defines

$$\|f\|_{\mathrm{BMO}^\beta} := \sup_Q \inf_{c \in \mathbb{R}} \frac{1}{\ell(Q)^\beta} \int_Q |f - c|\, d\mathcal{H}^\beta_\infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and the integral is understood in the sense of Choquet. When $\beta = n$, this definition recovers the classical BMO space up to equivalent seminorms, while for $\beta < n$ the spaces BMO^β are strictly smaller.

The first part of the chapter is devoted to the β-dimensional sharp maximal function. Motivated by the spaces BMO^β introduced in **[CS24]**, we define the operator $\mathcal{M}_\beta^\#$ as the natural sharp maximal function associated with Hausdorff content. We then prove a capacitary Fefferman–Stein inequality. The proof is based on a good-lambda inequality adapted to this setting, since the non-additivity of $\mathcal{H}_\infty^\beta$ prevents a direct use of the classical argument. We also obtain a capacitary analogue of the Muckenhoupt–Wheeden inequality for Riesz potentials. More precisely, for $\beta \in (n-\alpha, n]$, we prove that the $L^p(\mathcal{H}_\infty^\beta)$ norms of the Riesz potential $I_\alpha \mu$ and the fractional maximal function $\mathcal{M}_\alpha \mu$ are comparable.

We conclude the chapter with the study of commutators of fractional integrals. In the classical Lebesgue setting, Chanillo's theorem **[Cha82]** shows that the boundedness of the Riesz commutator $[b, I_\alpha] := b\, I_\alpha - I_\alpha(b\,\cdot)$ is equivalent to the condition $b \in \mathrm{BMO}$. More precisely, Chanillo proved that, for $1 < p < \frac{n}{\alpha}$ and q defined by $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$, with equivalence between the operator norm of the commutator and the BMO norm of the symbol. We prove an analogue of this characterization in the Hausdorff content setting, replacing BMO by BMO^β and the classical Lebesgue spaces by the corresponding capacitary spaces. We also prove an endpoint modular weak-type estimate for this commutator, extending the result of Cruz-Uribe and Fiorenza **[CF03]** to the framework of Hausdorff content.

Introducción

En esta tesis presento la investigación que he llevado a cabo durante mis cuatro años como estudiante de doctorado, desde septiembre de 2022 hasta julio de 2026, bajo la dirección de Carlos Pérez. El tema principal de esta tesis es el estudio de desigualdades de Poincaré–Sobolev con pesos, con especial énfasis en estimaciones cuantitativas, el fenómeno de automejora y pesos de Muckenhoupt. Sin embargo, durante estos años también he trabajado en algunos problemas relacionados, y otros menos relacionados, dentro del análisis armónico. Esta tesis pretende ser razonablemente autocontenida: el Capítulo 1 recoge la notación, las definiciones y los resultados conocidos que se utilizarán repetidamente a lo largo de este trabajo, mientras que cada uno de los capítulos posteriores contiene su propia introducción y motivación. En lo que resta de esta introducción, describimos brevemente los temas principales de la tesis capítulo por capítulo.

El Capítulo 2 está dedicado a propiedades de automejora de las desigualdades de Poincaré generalizadas. Esta línea de investigación tiene su origen en el trabajo de Franchi, Pérez y Wheeden **[FPW98]**. El punto de partida de este método es una desigualdad de Poincaré generalizada de la forma

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le a(Q),$$

para todo cubo Q, donde $f_Q = \frac{1}{|Q|}\int_Q f(y)\,dy$ y a es un funcional definido sobre cubos. Esta formulación modeliza varios tipos de desigualdades. Por ejemplo, si $f \in \mathrm{BMO}$, basta tomar $a(Q) = \|f\|_{\mathrm{BMO}}$ para cada cubo Q. También incluye la clásica desigualdad local de Poincaré $(1,p)$,

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le c_n\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

De manera similar, las desigualdades de Poincaré de orden superior también quedan cubiertas por este tipo de desigualdades, tras reemplazar el promedio f_Q por una aproximación polinómica adecuada $P_Q^m f$,

$$\frac{1}{|Q|}\int_Q |f(x)-P_Q^m f(x)|\,dx \le c_{n,m}\,\ell(Q)^m\left(\frac{1}{|Q|}\int_Q |\nabla^m f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

El mismo punto de vista también incluye desigualdades fraccionarias de Poincaré. Para $0<\delta<1$,

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le c_n(1-\delta)^{\frac{1}{p}}\,\ell(Q)^{\delta}\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}},$$

donde el factor $(1-\delta)^{\frac{1}{p}}$ es la ganancia extra procedente del fenómeno de Bourgain–Brezis–Mironescu **[BBM01]**.

La característica común detrás de estos ejemplos es que el funcional correspondiente satisface una condición geométrica de sumabilidad. Como veremos en este capítulo, esta condición es precisamente la que implica el fenómeno de automejora. Más concretamente, utilizaremos la condición $SD_p^s(w)$, introducida por Pérez y Rela **[PR19]**. Dado un peso w en $\mathbb{R}^n$, $s>0$ y $0<p<\infty$, decimos que el funcional a satisface la condición $SD_p^s(w)$ si existe una constante $C>0$ tal que, para todo cubo Q y toda familia $\{Q_j\}_j$ de subcubos disjuntos de Q, se tiene

$$\sum_j a(Q_j)^p w(Q_j) \leq C^p \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q).$$

Bajo esta condición, y suponiendo $w \in A_r$, probamos que la desigualdad inicial se automejora en el rango subcrítico $p<rs$. Más precisamente, si $p_{r,s}^*$ se define mediante la relación

$$\frac{1}{p} - \frac{1}{p_{r,s}^*} = \frac{1}{rs},$$

entonces obtenemos la siguiente desigualdad de Poincaré–Sobolev generalizada

$$\left\|f - f_Q\right\|_{L^{p_{r,s}^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \leq c_n\, p_{r,s}^*\, [w]_{A_\infty}\, [w]_{A_r}^{\frac{1}{rs}}\, \|a\|_{SD_p^s(w)}\, a(Q),$$

para todo cubo Q, donde $\|a\|_{SD_p^s(w)}$ es la mejor constante en la condición $SD_p^s(w)$. También estudiamos el caso crítico y supercrítico $p \geq rs$, donde la automejora correspondiente da integrabilidad exponencial en lugar de una estimación $L^{p_{r,s}^*,\infty}$.

La segunda parte del Capítulo 2 contiene aplicaciones y extensiones de este resultado abstracto. Primero presentamos tres extensiones del teorema general de automejora: a oscilaciones con respecto a proyecciones polinómicas, a funciones vectoriales y al contexto de los rectángulos. Las secciones restantes se centran en aplicaciones a desigualdades de Poincaré–Sobolev con pesos, desigualdades fraccionarias de Poincaré–Sobolev, incluyendo versiones con dos pesos, estimaciones exponenciales y desigualdades de Poincaré–Sobolev con normas de Lorentz del gradiente. Finalmente, mediante un método completamente diferente, estudiamos desigualdades de Poincaré–Sobolev en las que aparecen normas débiles en ambos lados de la desigualdad.

En el Capítulo 3, estudiamos desigualdades de Poincaré–Sobolev con pesos mediante integración fraccionaria. Este capítulo sigue un enfoque diferente del método de automejora estudiado en el Capítulo 2. Aunque dicho marco abstracto permite tratar muchas desigualdades de tipo Poincaré de manera unificada, en general no proporciona dependencia óptima respecto de las constantes características de los pesos. En el Capítulo 3, en lugar de partir de una desigualdad de Poincaré generalizada y mejorarla mediante una condición discreta de sumabilidad, utilizamos fórmulas de representación y estimaciones con pesos para el potencial de Riesz. La idea es clásica: se controla la oscilación de una función mediante el potencial de Riesz de su gradiente, y después se aplican estimaciones con pesos para el correspondiente operador potencial. Este enfoque, combinado con el método del truncamiento, nos

permite obtener desigualdades de Poincaré–Sobolev con pesos y con dependencia óptima respecto de la constante de Muckenhoupt.

Más específicamente, sean $n \geq 2$, $1 \leq p < n$, y sea $w \in A_r$ para algún $1 \leq r \leq p$. Si $r > 1$, consideramos el exponente q_r definido por

$$\frac{1}{p} - \frac{1}{q_r} = \frac{1}{n} \frac{\tau_n [\sigma]_{A_\infty}}{1 + r(\tau_n [\sigma]_{A_\infty} - 1)},$$

donde $\sigma = w^{1-r'}$, mientras que en el caso $r = 1$ tomamos $q_1 = p^* = \frac{np}{n-p}$. Probamos que existe $c_n > 0$ tal que

$$\left(\frac{1}{w(Q)} \int_Q |f(x) - f_Q|^{q_r} w(x)\, dx \right)^{\frac{1}{q_r}} \leq c_n p^* [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x)\, dx \right)^{\frac{1}{p}},$$

para cada cubo Q. Además, la potencia $\frac{1}{p}$ en la constante A_r es óptima. En particular, en el caso $r = 1$, esto da una respuesta positiva a una conjetura propuesta por Pérez y Rela en **[PR19]** sobre la potencia óptima de la constante A_1 en esta desigualdad.

También estudiamos el exponente óptimo que puede aparecer en el lado izquierdo de la desigualdad de Poincaré–Sobolev con pesos anterior para un peso fijo $w \in A_p$. Nos referimos a esta cantidad como el exponente de Sobolev óptimo asociado al peso w, y lo denotamos por p_w^*. Más precisamente, probamos que viene dado por la expresión

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n} \frac{1}{\ell_w},$$

donde

$$\ell_w := \inf\{r \geq 1 : \ w \in A_r\}.$$

Por tanto, dado un peso $w \in A_\infty$, estimar ℓ_w se convierte en un problema natural. Esto motiva el estudio cuantitativo de la inclusión de A_∞ en A_p para algún $p \geq 1$, al que está dedicada la parte final del capítulo. En particular, probamos un resultado de inmersión asintóticamente óptimo para pesos planos. Como consecuencia, obtenemos una nueva desigualdad de Poincaré–Sobolev para pesos A_∞ cuya constante de Fujii–Wilson es pequeña.

El Capítulo 4 está dedicado a estimaciones cuantitativas BMO–BLO para el operador maximal de Hardy–Littlewood. El punto de partida es el resultado clásico de Bennett, DeVore y Sharpley **[BDS81]**, donde se prueba que el operador maximal $\mathcal{M}$ está acotado en BMO. Esto fue posteriormente mejorado por Bennett **[Ben82]**, que probó una propiedad de acotación más fuerte, $\mathcal{M} : \mathrm{BMO} \to \mathrm{BLO}$, donde BLO está estrictamente contenido en BMO. Nuestro objetivo en este capítulo es estudiar esta acotación cuantitativamente y en el contexto con pesos. Una de las motivaciones principales proviene de la extensión de Karagulyan de la desigualdad de John–Nirenberg **[Kar02]**, donde la oscilación de una función localmente integrable no constante arbitraria se normaliza dividiendo por su función maximal sharp. Siguiendo esta idea, obtenemos estimaciones exponenciales para la oscilación inferior de $\mathcal{M}f$, normalizada por $\mathcal{M}^\# f$. Concretamente, si f es una función localmente integrable no constante tal

que $\mathcal{M}f$ no es idénticamente infinita y $w \in A_\infty$, entonces probamos que para todo cubo Q y todo $1 \le p < \infty$,

$$\left(\frac{1}{w(Q)}\int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^{\#}f(x)}\right)^p w(x)\,dx\right)^{\frac{1}{p}} \le c_n\,[w]_{A_\infty}\,p.$$

La dependencia lineal de p en el lado derecho es equivalente a decaimiento exponencial para la medida asociada al peso de los correspondientes conjuntos de nivel.

En la segunda parte del capítulo, estudiamos el recíproco, es decir, si la condición $w \in A_\infty$ es también necesaria para este tipo de estimación. Demostramos que, efectivamente, esto es así. Específicamente, probamos una nueva caracterización cuantitativa de la clase A_∞ en términos de la acotación

$$\mathcal{M} : \mathrm{BMO} \longrightarrow \mathrm{BLO}_w,$$

que resulta ser válida si y solo si $w \in A_\infty$. Esta caracterización cuantitativa muestra además que la dependencia lineal en $[w]_{A_\infty}$ en la estimación anterior es óptima.

En el Capítulo 5 probamos una fórmula puntual de Bourgain–Brezis–Mironescu (BBM) para potenciales de Riesz. Motivados por fórmulas recientes de subrepresentación establecidas en **[HMP25a]**, estudiamos el comportamiento límite, cuando $\alpha \to 1^-$, del operador potencial

$$(1-\alpha)I_\alpha(\mathcal{D}^\alpha f),$$

donde $\mathcal{D}^\alpha$ es el operador diferencial fraccionario no lineal introducido por Spector **[Spe20]**, definido por

$$\mathcal{D}^\alpha f(x) = \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dy, \qquad 0<\alpha<1.$$

Este operador puede interpretarse como un gradiente fraccionario no local de orden α, y el factor $(1-\alpha)$ es la normalización natural de BBM **[BBM01]**. El resultado principal del capítulo muestra que, para $f \in C_c^\infty(\mathbb{R}^n)$ y $n \ge 2$,

$$\lim_{\alpha\to 1^-}(1-\alpha)I_\alpha(\mathcal{D}^\alpha f)(x) = K_n I_1(|\nabla f|)(x),$$

para todo $x \in \mathbb{R}^n$, donde K_n es la misma constante geométrica que aparece en la fórmula clásica de BBM. Posteriormente, extendemos este resultado a funciones en el espacio de Sobolev $W^{1,1}(\mathbb{R}^n)$ mediante un argumento de densidad: dada cualquier sucesión $\alpha_k \to 1^-$, obtenemos convergencia en casi todo punto mediante una subsucesión.

El Capítulo 6 se dedica a un tema diferente al de los capítulos anteriores. En este capítulo, estudiamos análisis armónico respecto del contenido de Hausdorff, reemplazando la medida de Lebesgue por una capacidad, concretamente por la función de conjuntos no aditiva $\mathcal{H}^\beta_\infty$ definida por

$$\mathcal{H}^\beta_\infty(E) := \inf\left\{\sum_{j=1}^\infty \omega_\beta r_j^\beta : E \subset \bigcup_{j=1}^\infty B(x_j,r_j)\right\},$$

donde ω_β es una constante de normalización. Este punto de vista es natural en el estudio de cuestiones relacionadas con trazas, fenómenos de dimensión inferior e inmersiones de Sobolev en el endpoint. En

este contexto, un sustituto natural del espacio BMO viene dado por los espacios BMO^β, introducidos por Chen y Spector **[CS24]**, donde la oscilación se mide respecto de $\mathcal{H}^\beta_\infty$. Más precisamente, para $0 < \beta \le n$, se define

$$\|f\|_{\mathrm{BMO}^\beta} := \sup_Q \inf_{c\in\mathbb{R}} \frac{1}{\ell(Q)^\beta} \int_Q |f-c|\, d\mathcal{H}^\beta_\infty,$$

donde el supremo se toma sobre todos los cubos $Q \subset \mathbb{R}^n$, y la integral se entiende en el sentido de Choquet. Cuando $\beta = n$, esta definición recupera el espacio BMO salvo equivalencia de seminormas, mientras que para $\beta < n$ los espacios BMO^β son estrictamente más pequeños.

La primera parte del capítulo está dedicada a la función maximal sharp β-dimensional. Motivados por los espacios BMO^β introducidos en **[CS24]**, definimos el operador $\mathcal{M}^\#_\beta$ como la función maximal sharp natural asociada al contenido de Hausdorff. Después probamos una desigualdad capacitaria de Fefferman–Stein. La demostración se basa en una desigualdad de tipo good-lambda adaptada a este contexto, ya que la no aditividad de $\mathcal{H}^\beta_\infty$ impide utilizar el argumento clásico. También obtenemos un análogo capacitario de la desigualdad de Muckenhoupt–Wheeden. Concretamente, para $\beta \in (n-\alpha, n]$, probamos que las normas $L^p(\mathcal{H}^\beta_\infty)$ del potencial de Riesz $I_\alpha\mu$ y de la función maximal fraccionaria $\mathcal{M}_\alpha\mu$ son comparables.

Concluimos el capítulo con el estudio de conmutadores de integrales fraccionarias. En el contexto clásico con la medida de Lebesgue, el teorema de Chanillo **[Cha82]** muestra que la acotación del conmutador de Riesz $[b, I_\alpha] := b\, I_\alpha - I_\alpha(b\,\cdot)$ es equivalente a la condición $b \in \mathrm{BMO}$. Específicamente, Chanillo probó que, para $1 < p < \frac{n}{\alpha}$ y q definido por $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, el conmutador $[b, I_\alpha]$ está acotado de $L^p(\mathbb{R}^n)$ en $L^q(\mathbb{R}^n)$ si y solo si $b \in \mathrm{BMO}(\mathbb{R}^n)$, con equivalencia entre la norma de operador del conmutador y la seminorma BMO del símbolo. Probamos un análogo de esta caracterización en el contexto del contenido de Hausdorff, reemplazando BMO por BMO^β y los espacios clásicos de Lebesgue por los correspondientes espacios capacitarios. También probamos en el endpoint una estimación modular débil para este conmutador, extendiendo el resultado de Cruz-Uribe y Fiorenza **[CF03]** al marco del contenido de Hausdorff.

Chapter 1

Preliminaries

In this chapter, we establish the notation, definitions, and background results that will be used throughout the thesis. We begin by fixing the geometric and measure-theoretic conventions, and then recall the main function spaces appearing in the sequel, including Lebesgue, Lorentz, Sobolev, fractional Sobolev, and BMO spaces. We also review the Hardy–Littlewood maximal operator, Muckenhoupt weights, fractional operators, and the quantitative properties of A_p and A_∞ weights that will be needed later. Finally, we present the Poincaré and Poincaré–Sobolev inequalities, together with the truncation methods used to upgrade weak-type estimates to stronger Lorentz-scale bounds.

1.1 Geometric and measure-theoretic conventions

Throughout this thesis, unless otherwise stated, we work in the Euclidean space $\mathbb{R}^n$ with $n \geq 1$. A cube Q is a subset of $\mathbb{R}^n$ that admits an expression as a Cartesian product of n intervals of the same length, meaning its sides are parallel to the coordinate axes. This common length is called the side length of Q, denoted by $\ell(Q)$. If these intervals are all open, then the cube is called open, and if they are all closed, then the cube is called closed. We denote the collection of all such cubes by $\mathcal{Q}$. For a constant $c > 0$, we denote by cQ the cube concentric with Q whose side length is $c\ell(Q)$. In particular, $2Q$ is the cube with the same center as Q and double the side length.

We denote by $\mathcal{D}$ the standard dyadic lattice in $\mathbb{R}^n$, which consists of all cubes of the form

$$2^{-k}([0,1)^n + m)$$

for $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. More generally, given a cube Q, we denote by $\mathcal{D}_Q$ the dyadic lattice generated by Q, that is, the lattice obtained by translating and dilating the standard dyadic lattice so that Q is one of its cubes. Furthermore, we denote by $\mathcal{D}(Q)$ the collection of dyadic subcubes of Q. This collection consists of Q itself and all the subcubes obtained by repeatedly bisecting Q and its descendants into 2^n subcubes of half the side length.

When more convenient, we will also work with Euclidean balls, denoting by $B(x, r)$ the ball centered at x with radius r. Similarly to cubes, for a constant $c > 0$, cB denotes the ball concentric with B and radius cr. For a point $x \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$, the Euclidean distance from x to A is defined as

$$\operatorname{dist}(x, A) = \inf_{y \in A} |x - y|.$$

Regarding measure theory, let us recall some standard definitions. Let X be a set equipped with a σ-algebra Σ. A set function $\mu : \Sigma \to [0, \infty]$ is called a measure if it satisfies the following properties:

1. $\mu(\emptyset) = 0$.

2. For any countable collection $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in Σ, we have

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i).$$

If these conditions are satisfied, the triplet (X, Σ, μ) is called a measure space. Furthermore, when X is a topological space (such as $\mathbb{R}^n$), the σ-algebra generated by the open sets of X is called the Borel σ-algebra, denoted by $\mathcal{B}(X)$. A measure μ defined on the measurable space $(X, \mathcal{B}(X))$ is called a Borel measure. In the particular case of $\mathbb{R}^n$, a Borel measure μ is called a Radon measure if it is finite on compact sets. In the Euclidean setting, this condition implies the following regularity properties: the measure of every Borel set can be approximated from inside by compact sets and from outside by open sets (see **[Tor88]**).

Focusing on $\mathbb{R}^n$, we denote the Lebesgue measure of a Lebesgue measurable set $E \subset \mathbb{R}^n$ by $|E|$. Furthermore, when a set or a function is said to be measurable without explicit reference to a measure, it is understood to be Lebesgue measurable. Similarly, the abbreviation "a.e." (almost everywhere) without specifying a measure refers to Lebesgue almost everywhere.

We say that a Borel measure μ on $\mathbb{R}^n$ is doubling if there exists a constant $D_\mu > 0$ such that, for every cube $Q \subset \mathbb{R}^n$,

$$\mu(2Q) \leq D_\mu \mu(Q).$$

The Lebesgue measure is a standard example of a doubling measure, since $|2Q| = 2^n |Q|$, with doubling constant 2^n.

We will frequently deal with integral averages. Let μ be a Borel measure on $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be a μ-measurable set such that $0 < \mu(E) < \infty$. For any function f that is integrable on E with respect to μ, we denote the integral average of f over E by

$$f_{E,\mu} := \frac{1}{\mu(E)} \int_E f \, d\mu.$$

In the particular case where μ is the Lebesgue measure, we omit the measure subscript and simply write

$$f_E := \fint_E f(x) \, dx = \frac{1}{|E|} \int_E f(x) \, dx.$$

Furthermore, for any subset $E \subset \mathbb{R}^n$, we denote its characteristic function (or indicator function) by χ_E, which is defined as

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

As is standard in harmonic analysis, the letter C (or c) will denote various positive constants whose exact values may change from line to line, even within the same string of estimates. When we write $C(\alpha, \beta, \dots)$ or $C_{\alpha,\beta,\dots}$ (and similarly for c), we indicate that the constant depends only on the parameters $\alpha, \beta, \dots$.

1.2 Function spaces

In this section, we review the basic definitions and properties of several function spaces that will be used throughout the thesis. We first introduce the classical Lebesgue spaces and then the Lorentz spaces. Although Lebesgue spaces are a particular case of the more general Lorentz scale, we present them in separate subsections for the sake of clarity. We then briefly recall the definitions of Sobolev and fractional Sobolev spaces, as well as the space of bounded mean oscillation BMO, which will play an important role in several of the subsequent chapters.

Throughout this thesis, unless otherwise stated, functions are assumed to be defined on $\mathbb{R}^n$ and take real or complex values.

1.2.1 Lebesgue spaces

Let us begin by recalling the formal definition of the classical Lebesgue spaces over a general measure space.

Definition 1.1. Let (X, Σ, μ) be a measure space and let $0 < p \leq \infty$. The Lebesgue space $L^p(X, \mu)$ is defined as the set of all measurable functions $f : X \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) for which the quantity $\|f\|_{L^p(X,\mu)}$ is finite, where

$$\|f\|_{L^p(X,\mu)} := \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

if $0 < p < \infty$, and for the endpoint $p = \infty$,

$$\|f\|_{L^\infty(X,\mu)} := \operatorname*{ess\,sup}_{x \in X} |f(x)| = \inf\{t > 0 : \mu(\{x \in X : |f(x)| > t\}) = 0\}.$$

As usual, we identify functions that are equal μ-almost everywhere. To simplify the notation, when the measure is the Lebesgue measure, we will write $L^p(X)$ and $\|f\|_{L^p(X)}$. Furthermore, when the underlying space is $\mathbb{R}^n$, we will also write $L^p(\mu)$ and $\|f\|_{L^p(\mu)}$. Thus, we will simply write L^p instead of $L^p(\mathbb{R}^n, dx)$, and denote its corresponding norm by $\|f\|_{L^p}$.

We say that a measurable function $f : \Omega \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) on an open set $\Omega \subset \mathbb{R}^n$ is locally integrable, and we write $f \in L^1_{\text{loc}}(\Omega)$, if $f \in L^1(K)$ for every compact subset $K \subset \Omega$. Following our previous conventions, when the underlying space is $\mathbb{R}^n$, we will simply write L^1_{loc} instead of $L^1_{\text{loc}}(\mathbb{R}^n)$.

The structural properties of $L^p(X,\mu)$ depend heavily on the parameter p. When $1 \le p \le \infty$, the functional $\|\cdot\|_{L^p(X,\mu)}$ is a norm. In particular, it satisfies the triangle inequality, which in this context is known as Minkowski's inequality:

$$\|f+g\|_{L^p(X,\mu)} \le \|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)}.$$

Equipped with this norm, $L^p(X,\mu)$ is a complete metric space, making it a Banach space.

A fundamental tool in harmonic analysis is Hölder's inequality. For any exponent $1 \le p \le \infty$, we define its conjugate exponent $p' \in [1,\infty]$ by the algebraic relation

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

under the standard conventions that $1' = \infty$ and $\infty' = 1$. Hölder's inequality states that if $f \in L^p(X,\mu)$ and $g \in L^{p'}(X,\mu)$, then their product fg belongs to $L^1(X,\mu)$, and

$$\int_X |f(x)g(x)|\, d\mu(x) \le \|f\|_{L^p(X,\mu)}\, \|g\|_{L^{p'}(X,\mu)}.$$

Finally, we record the "layer cake" formula, which is an identity for computing the L^p norm of a function in terms of its distribution function. For any $0 < p < \infty$ and any measurable function f, Fubini's theorem yields

$$\int_X |f(x)|^p\, d\mu(x) = p\int_0^\infty t^{p-1}\mu\big(\{x \in X : |f(x)| > t\}\big)\, dt.$$

1.2.2 Lorentz spaces

We briefly recall the definition and some basic properties of Lorentz spaces. Following our conventions, let (X,Σ,μ) be a measure space. For $0 < p < \infty$ and $0 < q \le \infty$, the Lorentz space $L^{p,q}(X,\mu)$ consists of all measurable functions f such that the quantity

$$\|f\|_{L^{p,q}(X,\mu)} := \begin{cases} \left(p\displaystyle\int_0^\infty t^q\, \mu\big(\{x \in X :\ |f(x)| > t\}\big)^{\frac{q}{p}}\, \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty, \\ \displaystyle\sup_{t>0} t\, \mu\big(\{x \in X :\ |f(x)| > t\}\big)^{\frac{1}{p}}, & q = \infty, \end{cases} \tag{1.1}$$

is finite. The functional $\|\cdot\|_{L^{p,q}(X,\mu)}$ is a norm if and only if either $1 \le q \le p$ or $p = q = \infty$; otherwise it is a quasi-norm. In any case, it satisfies the quasi-triangle inequality

$$\|f+g\|_{L^{p,q}(X,\mu)} \le c_{p,q}\big(\|f\|_{L^{p,q}(X,\mu)} + \|g\|_{L^{p,q}(X,\mu)}\big).$$

Moreover, if $1 < p < \infty$ and $1 \le q \le \infty$ (as well as in the endpoint cases $p = q = 1$ and $p = q = \infty$), the quasi-norm (1.1) is equivalent to a true norm (see **[BS88]**).

Remark 1.2. Note that we include the normalization factor p in the definition of the Lorentz quasi-norm (1.1). While other equivalent normalizations sometimes appear in the literature, this specific choice ensures that by the layer cake formula, we have $\|f\|_{L^{p,p}(X,\mu)} = \|f\|_{L^p(X,\mu)}$ for every $0 < p < \infty$.

We will use the following form of Hölder's inequality in Lorentz spaces: if $1 \le p < \infty$, $1 \le q \le \infty$, and $f \in L^{p,q}(X,\mu)$, $g \in L^{p',q'}(X,\mu)$, then

$$\int_X |f(x)g(x)|\, d\mu(x) \le \|f\|_{L^{p,q}(X,\mu)}\, \|g\|_{L^{p',q'}(X,\mu)}. \tag{1.2}$$

We also recall the nestedness of the scale with respect to the second index (see **[BS88]**): if $0 < p \le \infty$ and $0 < q_1 \le q_2 \le \infty$, then

$$\|f\|_{L^{p,q_2}(X,\mu)} \le \left(\frac{q_1}{p}\right)^{\frac{1}{q_1}-\frac{1}{q_2}} \|f\|_{L^{p,q_1}(X,\mu)}. \tag{1.3}$$

Finally, we recall a standard embedding between Lorentz spaces on a probability space (see **[Hun66]**). Let (X,ν) be a measure space with $\nu(X) = 1$. If $0 < p_1 < p_2 < \infty$ and $0 < q < \infty$, then

$$\|f\|_{L^{p_1,q}(X,\nu)} \le \left(\frac{p_1 p_2}{q(p_2-p_1)}\right)^{\frac{1}{q}} \|f\|_{L^{p_2,\infty}(X,\nu)}. \tag{1.4}$$

An important and highly useful particular case of (1.4) arises when we set the second index to match the first, $q = p_1$. By taking $p_1 = r$ and $p_2 = p$, we recover the well-known Kolmogorov's inequality for probability spaces. Specifically, if $0 < r < p < \infty$, then for any $f \in L^{p,\infty}(X,\nu)$, we have $f \in L^r(X,\nu)$ and

$$\|f\|_{L^r(X,\nu)} \le \left(\frac{p}{p-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}(X,\nu)}. \tag{1.5}$$

We will need a geometric summation property that Lorentz quasi-norms satisfy, in the spirit of a result from **[CHK82]**.

Lemma 1.3. *Let (X,μ) be a measure space. Let $1 \le p, q < \infty$ and $\alpha \ge \max\{p,q\}$, then we have*

$$\sum_j \|f\|^{\alpha}_{L^{p,q}(E_j,\mu)} \le \|f\|^{\alpha}_{L^{p,q}\left(\cup_j E_j,\mu\right)},$$

for each pairwise disjoint family $\{E_j\}_j$ of μ-measurable sets.

Proof. Let $\{E_j\}_j$ be a pairwise disjoint family of μ-measurable sets. Since $\frac{\alpha}{q} \ge 1$ we can use Minkowski's integral inequality,

$$\begin{aligned}
\left(\sum_j \|f\|^{\alpha}_{L^{p,q}(E_j,\mu)}\right)^{\frac{q}{\alpha}} &= \left(\sum_j \left(p\int_0^\infty t^q \mu\left(\{x\in E_j : |f(x)|>t\}\right)^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{\alpha}{q}}\right)^{\frac{q}{\alpha}} \\
&= \left\| \left\{ p\int_0^\infty t^q \mu\left(\{x\in E_j : |f(x)|>t\}\right)^{\frac{q}{p}} \frac{dt}{t} \right\}_j \right\|_{\ell^{\frac{\alpha}{q}}} \\
&\le p\int_0^\infty t^q \left\| \left\{ \mu\left(\{x\in E_j : |f(x)|>t\}\right)^{\frac{q}{p}} \right\}_j \right\|_{\ell^{\frac{\alpha}{q}}} \frac{dt}{t} \\
&= p\int_0^\infty t^q \left\| \left\{ \mu\left(\{x\in E_j : |f(x)|>t\}\right) \right\}_j \right\|^{\frac{q}{p}}_{\ell^{\frac{\alpha}{p}}} \frac{dt}{t} \\
&\le p\int_0^\infty t^q \left\| \left\{ \mu\left(\{x\in E_j : |f(x)|>t\}\right) \right\}_j \right\|^{\frac{q}{p}}_{\ell^{1}} \frac{dt}{t}
\end{aligned}$$

$$
\begin{aligned}
&= p \int_0^\infty t^q \mu\left(\left\{x \in \cup_j E_j : |f(x)| > t\right\}\right)^{\frac{q}{p}} \frac{dt}{t} \\
&= \|f\|^q_{L^{p,q}\left(\cup_j E_j, \mu\right)},
\end{aligned}
$$

where in the two inequalities above we have used that $\frac{\alpha}{q} \geq 1$ and $\frac{\alpha}{p} \geq 1$ respectively. □

1.2.3 Sobolev spaces

In this brief section, we recall the standard definitions of classical and fractional Sobolev spaces on an open set $\Omega \subset \mathbb{R}^n$. Firstly, we need the concept of weak derivatives.

Definition 1.4. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in L^1_{\text{loc}}(\Omega)$. For an integer $i \in \{1, \ldots, n\}$, a function $g_i \in L^1_{\text{loc}}(\Omega)$ is called the i-th weak partial derivative of f if it satisfies the integration by parts formula

$$
\int_\Omega f(x) \partial_i \varphi(x)\, dx = -\int_\Omega g_i(x) \varphi(x)\, dx,
$$

for all test functions $\varphi \in C^\infty_c(\Omega)$. In this case, we write $\partial_i f = g_i$.

When all first-order weak partial derivatives exist, they form the weak gradient of f, denoted by the vector field

$$
\nabla f = (\partial_1 f, \ldots, \partial_n f).
$$

Higher-order weak derivatives are defined analogously by iterating this procedure. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ of order $|\alpha| = m \geq 1$, we denote the corresponding weak derivative by $D^\alpha f$. We then set

$$
|\nabla^m f| = \sum_{|\alpha| = m} |D^\alpha f|.
$$

We can now define the classical Sobolev spaces, originally introduced by Sobolev in **[Sob38]**. For $1 \leq p \leq \infty$, the Sobolev space $W^{1,p}(\Omega)$ is defined as the space of all functions $f \in L^p(\Omega)$ such that its weak gradient ∇f exists and belongs to $L^p(\Omega)$. It is a Banach space when equipped with the norm

$$
\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.
$$

A fundamental property of Sobolev spaces is the density of smooth functions. By the classical Meyers–Serrin theorem **[MS64]**, the space $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$. Furthermore, if we consider the whole space $\mathbb{R}^n$, the space of smooth functions with compact support $C^\infty_c(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$ (see for example **[AF03; Bre11]**). For a general open set Ω, the closure of $C^\infty_c(\Omega)$ with respect to the $W^{1,p}(\Omega)$ norm defines the subspace $W^{1,p}_0(\Omega)$. On sufficiently regular domains, this space can be interpreted as the class of Sobolev functions with zero trace on the boundary $\partial\Omega$. For a more comprehensive treatment of classical Sobolev spaces, including embedding theorems and trace operators, we refer to the monographs **[Maz85; AH96; AF03; Leo17]**.

To quantify intermediate degrees of smoothness, we introduce the fractional Sobolev spaces. These function spaces are often referred to as Aronszajn, Gagliardo, or Slobodeckij spaces, in recognition of

their almost simultaneous introduction by these authors **[Aro55; Gag58; Slo58]**. For $0 < \delta < 1$ and $1 \leq p < \infty$, the Gagliardo seminorm of a measurable function $f : \Omega \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is defined as

$$[f]_{W^{\delta,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}} \, dx \, dy \right)^{\frac{1}{p}}.$$

The fractional Sobolev space $W^{\delta,p}(\Omega)$ consists of all functions $f \in L^p(\Omega)$ for which this seminorm is finite. It is a Banach space when endowed with the norm

$$\|f\|_{W^{\delta,p}(\Omega)} := \|f\|_{L^p(\Omega)} + [f]_{W^{\delta,p}(\Omega)}.$$

As in the first-order case, smooth functions are dense in fractional Sobolev spaces. Specifically, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{\delta,p}(\mathbb{R}^n)$ for any $1 \leq p < \infty$ (see for instance **[AF03]**). For further details and an extensive overview of the theory of fractional Sobolev spaces, including their extension properties and fractional inequalities, we refer to the survey **[DPV12]** and to the recent monograph **[Leo23]**.

1.2.4 Bounded Mean Oscillation space

To conclude our review of function spaces, we introduce the space of Bounded Mean Oscillation (BMO). Originally introduced by John and Nirenberg in **[JN61]**, this space plays an important role in harmonic analysis, often serving as the natural substitute for L^∞ in the theory of Calderón–Zygmund operators.

Definition 1.5. A locally integrable function f is said to be of bounded mean oscillation, and we write $f \in \mathrm{BMO}(\mathbb{R}^n)$, if its BMO seminorm is finite,

$$\|f\|_{\mathrm{BMO}} := \sup_{Q \in Q} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty.$$

It is straightforward to check that $\|f\|_{\mathrm{BMO}} = 0$ if and only if f is equal to a constant almost everywhere. Therefore, $\mathrm{BMO}(\mathbb{R}^n)$ becomes a Banach space when functions differing by a constant are identified.

The average f_Q minimizes the L^p oscillation up to a factor of 2, allowing us to easily bound the oscillation by replacing f_Q with any convenient constant.

Proposition 1.6. *Let $1 \leq p < \infty$ and let $Q \subset \mathbb{R}^n$ be a cube. For any $f \in L^p(Q)$, we have the equivalence*

$$\inf_{c \in \mathbb{R}} \left(\frac{1}{|Q|} \int_Q |f(x) - c|^p \, dx \right)^{\frac{1}{p}} \leq \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{\frac{1}{p}} \leq 2 \inf_{c \in \mathbb{R}} \left(\frac{1}{|Q|} \int_Q |f(x) - c|^p \, dx \right)^{\frac{1}{p}}.$$

Proof. The first inequality holds trivially since $f_Q \in \mathbb{R}$. For the second inequality, let $c \in \mathbb{R}$ be arbitrary. By Minkowski's and Jensen's inequalities, we have

$$\begin{aligned}
\left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{\frac{1}{p}} &\leq \left(\frac{1}{|Q|} \int_Q |f(x) - c|^p \, dx \right)^{\frac{1}{p}} + |c - f_Q| \\
&\leq \left(\frac{1}{|Q|} \int_Q |f(x) - c|^p \, dx \right)^{\frac{1}{p}} + \frac{1}{|Q|} \int_Q |c - f(y)| \, dy \\
&\leq 2 \left(\frac{1}{|Q|} \int_Q |f(x) - c|^p \, dx \right)^{\frac{1}{p}}.
\end{aligned}$$

Taking the infimum over all $c \in \mathbb{R}$ yields the upper bound, completing the proof. □

A fundamental property of BMO functions is the exponential decay of their distribution function, a fact known as the John–Nirenberg theorem **[JN61]**.

Theorem 1.7. *There exist positive constants c_1 and c_2, depending only on the dimension n, such that for every $f \in \mathrm{BMO}(\mathbb{R}^n)$, any cube $Q \subset \mathbb{R}^n$, and any $\lambda > 0$,*

$$\left|\{x \in Q : |f(x) - f_Q| > \lambda\}\right| \leq c_1 |Q| \exp\left(-\frac{c_2 \lambda}{\|f\|_{\mathrm{BMO}}}\right).$$

A direct consequence of Theorem 1.7, which can be obtained by combining the exponential decay with the layer cake representation, is that the L^1 average in the definition of BMO can be replaced by any L^p average, with a very precise control over the constants. Specifically, for any $1 \leq p < \infty$, we have

$$\|f\|_{\mathrm{BMO}} \leq \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx\right)^{\frac{1}{p}} \leq C_n \, p \, \|f\|_{\mathrm{BMO}},$$

where $C_n > 0$ is a constant depending only on the dimension n. To the best of our knowledge, the best currently known dimensional constant in the upper estimate is $C_n = \sqrt{n}$, due to Wik **[Wik90]**. In fact, the relationship between these two properties is bidirectional. The linear dependence on p in the upper bound is not merely a consequence of the John–Nirenberg theorem, but it actually implies the exponential decay (see Proposition 4.4).

1.3 Maximal operators and weights

In this section, we review two objects that will play a key role throughout this thesis: the Hardy–Littlewood maximal operator and Muckenhoupt weights. After recalling the standard definitions of the maximal operator, we introduce the Muckenhoupt weight classes, which precisely characterize its boundedness on weighted Lebesgue spaces. Finally, we detail their core properties, including sharp reverse Hölder inequalities and the precise left-openness property.

1.3.1 Hardy–Littlewood maximal operators

For the remainder of this section, and indeed for most of the subsequent chapters, we will work exclusively with the Lebesgue measure on $\mathbb{R}^n$, unless explicitly stated otherwise.

Definition 1.8. Given a locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the (uncentered) Hardy–Littlewood maximal operator is defined by

$$\mathcal{M}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes such that $x \in Q$. The centered maximal operator is defined by

$$\mathcal{M}^c f(x) = \sup_{r > 0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)| \, dy,$$

where $Q(x, r)$ denotes the open cube centered at x with side length r.

Remark 1.9. Since the Lebesgue measure is doubling, $\mathcal{M}$ and $\mathcal{M}^c$ are pointwise comparable. Specifically, there exists a dimensional constant $c_n \geq 1$ such that

$$\mathcal{M}^c f(x) \leq \mathcal{M} f(x) \leq c_n \mathcal{M}^c f(x),$$

for every $x \in \mathbb{R}^n$ and every locally integrable function f. Consequently, any boundedness estimate for one operator implies the corresponding estimate for the other, up to dimensional constants. This pointwise equivalence relies on the doubling condition: if the underlying measure is a general non-doubling Borel measure, $\mathcal{M}$ and $\mathcal{M}^c$ are no longer pointwise equivalent, and their mapping properties may be different (see **[Sjö83]**).

An immediate consequence of the Lebesgue differentiation theorem is that these maximal operators control the size of the function pointwise. Specifically, for any locally integrable function f, we have

$$|f(x)| \leq \mathcal{M}^c f(x) \leq \mathcal{M} f(x),$$

for almost every $x \in \mathbb{R}^n$.

The Hardy–Littlewood maximal operators can also be defined by taking the supremum over Euclidean balls instead of cubes. In the case of the Lebesgue measure on $\mathbb{R}^n$, these definitions are pointwise equivalent, with constants depending only on the dimension. Therefore, any mapping property proved for one of these variants immediately yields the corresponding property for the other, up to dimensional constants.

By the classical Hardy–Littlewood maximal theorem, $\mathcal{M}$ (and thus $\mathcal{M}^c$) satisfies a weak type $(1,1)$ estimate, meaning that $\mathcal{M} : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ is bounded. Interpolating this with the trivial $L^\infty \to L^\infty$ bound yields that $\mathcal{M}$ is of strong type (p,p) for all $1 < p \leq \infty$, meaning that $\mathcal{M} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded.

A standard way to simplify the analysis of maximal operators is to restrict the supremum to dyadic cubes. This leads to dyadic variants of the Hardy–Littlewood maximal operator, which are technically more manageable and retain the essential features of the original operator.

Definition 1.10. The standard dyadic maximal operator is defined by

$$\mathcal{M}_d f(x) = \sup_{\substack{Q \in \mathcal{D} \\ Q \ni x}} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where $\mathcal{D}$ is the standard dyadic lattice in $\mathbb{R}^n$. Furthermore, given a fixed cube $Q_0 \in \mathcal{Q}$, we define the local dyadic maximal operator with respect to Q_0, for any $x \in Q_0$, as

$$\mathcal{M}_{Q_0} f(x) = \sup_{\substack{Q \in \mathcal{D}(Q_0) \\ Q \ni x}} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

It is clear that $\mathcal{M}_d f(x) \leq \mathcal{M} f(x)$ pointwise. The reverse inequality does not hold pointwise, but $\mathcal{M}$ can be controlled by a finite sum of shifted dyadic maximal operators. The classical "3^n-lattice theorem" states that any cube in $\mathbb{R}^n$ is contained in a dyadic cube from one of 3^n translated dyadic grids, with a side length comparable to the original cube. This geometric covering property implies

that $\mathcal{M}$ is pointwise bounded by the sum of 3^n dyadic maximal operators. The exact historical origin of this result is somewhat unclear. According to the historical discussion in **[Cru16]**, credit should be given to Okikiolu **[Oki92]**, while a weaker form had already appeared in the work of Chang, Wilson, and Wolff **[CWW85]**. Later, Conde **[Con13, Theorem A]** proved that the number of required dyadic lattices can be reduced to just $n + 1$, and he showed that this number is sharp. We will utilize a refined version of this result (we refer to **[Can21, Lemma 4.34]** for the proof of this specific statement).

Lemma 1.11. *Let $Q \subset \mathbb{R}^n$ be a cube. There exist $n+1$ dyadic grids $\{\mathcal{D}_k\}_{k=0}^n$ and $n+1$ cubes $Q_k \in \mathcal{D}_k$ satisfying the following conditions:*

1. *Given any cube $R \subset \mathbb{R}^n$, there exists $k \in \{0, \dots, n\}$ and $P \in \mathcal{D}_k$ such that $R \subset P$ and $\ell(P) \leq 3\ell(R)$. Consequently,*

$$\mathcal{M}f(x) \simeq_n \sum_{k=0}^{n} \mathcal{M}_d^{(k)} f(x), \tag{1.6}$$

   where $\mathcal{M}_d^{(k)}$ denotes the dyadic maximal function with respect to the grid $\mathcal{D}_k$.

2. *$Q \subset \bigcap_{k=0}^n Q_k$ and $\ell(Q) \simeq \ell(Q_k)$ for all k.*

A related operator is the sharp maximal operator. Given a locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the sharp maximal function is defined by

$$\mathcal{M}^\# f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and containing the point x.

The sharp maximal function is related to BMO. In fact, a function f belongs to BMO if and only if $\mathcal{M}^\# f$ is bounded, and the BMO seminorm is precisely given by the L^∞ norm of the sharp maximal function: $\|f\|_{\text{BMO}} = \|\mathcal{M}^\# f\|_{L^\infty(\mathbb{R}^n)}$.

Furthermore, by a simple application of the triangle inequality, it is easy to see that the sharp maximal function is pointwise controlled by the standard Hardy–Littlewood maximal function. Indeed, for any cube Q with $x \in Q$, we have

$$\frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \leq \frac{1}{|Q|} \int_Q |f(y)| \, dy + |f_Q| \leq \frac{2}{|Q|} \int_Q |f(y)| \, dy \leq 2\mathcal{M}f(x),$$

which yields the pointwise bound $\mathcal{M}^\# f(x) \leq 2\mathcal{M}f(x)$ for all $x \in \mathbb{R}^n$. While the reverse inequality trivially fails pointwise (for instance, if f is a non-zero constant, $\mathcal{M}^\# f = 0$ but $\mathcal{M}f = |f| > 0$), Fefferman and Stein demonstrated in their celebrated work **[FS71]** that these two operators are equivalent in norm. Specifically, for any $1 < p < \infty$ and any $f \in L^p(\mathbb{R}^n)$, we have

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leq c_{n,p} \|\mathcal{M}^\# f\|_{L^p(\mathbb{R}^n)}. \tag{1.7}$$

1.3.2 Muckenhoupt weights

We now recall the Muckenhoupt classes, which characterize the weighted boundedness of the Hardy–Littlewood maximal operator. By a weight w, we mean a non-negative locally integrable function on $\mathbb{R}^n$ that is strictly positive and finite almost everywhere. Given a measurable set $E \subset \mathbb{R}^n$, we denote its w-measure by $w(E) = \int_E w(x)\,dx$.

A weight w is said to belong to the Muckenhoupt class A_p for $1 < p < \infty$ if the quantity

$$[w]_{A_p} := \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'}\,dx \right)^{p-1}$$

is finite. For the endpoint $p = 1$, we write $w \in A_1$ if

$$[w]_{A_1} := \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \operatorname*{ess\,sup}_{x \in Q} w(x)^{-1}$$

is finite. Equivalently, the A_1 condition can be restated pointwise as

$$\mathcal{M}w(x) \leq [w]_{A_1}\, w(x) \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

We will frequently use the following two elementary properties of the A_p classes, both of which are direct consequences of the definition and Jensen's inequality (see **[GF85]**). First, the A_p classes are nested, and their constants are monotonically decreasing with respect to p. Specifically, if $1 \leq p \leq q < \infty$, then $A_p \subset A_q$ and

$$[w]_{A_q} \leq [w]_{A_p}.$$

Second, for $1 < p < \infty$, the A_p condition possesses a symmetry with respect to the dual exponent p'. A weight w belongs to A_p if and only if its associated dual weight, defined as $\sigma = w^{1-p'}$, belongs to $A_{p'}$. Moreover, their constants are precisely related by the identity

$$[w]_{A_p} = [\sigma]_{A_{p'}}^{p-1}.$$

We will frequently use the following property of A_p weights: a weight w belongs to the class A_p if and only if for every measurable function $f \geq 0$, we have

$$\frac{1}{|Q|} \int_Q f(x)\,dx \leq [w]_{A_p}^{\frac{1}{p}} \left(\frac{1}{w(Q)} \int_Q f(x)^p w(x)\,dx \right)^{\frac{1}{p}}, \tag{1.8}$$

for every cube Q. In particular, taking $f = \chi_E$ for a measurable set $E \subset \mathbb{R}^n$, we obtain

$$\frac{|E|}{|Q|} \leq [w]_{A_p}^{\frac{1}{p}} \left(\frac{w(E)}{w(Q)} \right)^{\frac{1}{p}}. \tag{1.9}$$

We also define the limiting class

$$A_\infty := \bigcup_{p \geq 1} A_p.$$

The membership to A_∞ can also be characterized in terms of the finiteness of the so-called A_∞ constant defined as

$$[w]'_{A_\infty} := \sup_{Q} \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \exp\left(\frac{1}{|Q|} \int_Q \log w(x)^{-1}\,dx \right) \tag{1.10}$$

that was originally introduced by Hruščev **[Hru84]** (see also **[GF85]**). In recent years, a new definition was introduced that is more adequate to express quantitative norm inequalities for maximal and singular operators, namely the one defined as

$$[w]_{A_\infty} = \sup_{Q\in\mathcal{Q}} \frac{1}{w(Q)} \int_Q \mathcal{M}(w\chi_Q)(x)\,dx.$$

This is sometimes called the Fujii–Wilson constant, as it was implicitly introduced by Wilson a long time ago (see **[Wil87; Wil89; Wil08]**) with a different notation, and was also implicitly introduced by Fujii **[Fuj78]**. Its explicit formulation appears in **[HP13]**. This constant is more relevant since it is known that $[w]'_{A_\infty} < \infty$ if and only if $[w]_{A_\infty} < \infty$, but there are examples of weights $w \in A_\infty$ so that $[w]_{A_\infty}$ is much smaller than $[w]'_{A_\infty}$ (interesting examples can be found in **[HP13]**, where it was shown that $[w]'_{A_\infty}$ can be exponentially larger than $[w]_{A_\infty}$).

Muckenhoupt **[Muc72]** characterized the weights for which $\mathcal{M}$ is bounded on weighted Lebesgue spaces: for $1 \le p < \infty$, the weak-type estimate

$$\mathcal{M} : L^p(w) \longrightarrow L^{p,\infty}(w)$$

holds if and only if $w \in A_p$; and for $1 < p < \infty$, the strong-type bound

$$\mathcal{M} : L^p(w) \longrightarrow L^p(w)$$

holds if and only if $w \in A_p$.

These operator bounds can be quantified in terms of the A_p constants. For the weak-type inequality, it is known that for $p \ge 1$,

$$\|\mathcal{M}\|_{L^p(w)\longrightarrow L^{p,\infty}(w)} \simeq_n [w]_{A_p}^{\frac{1}{p}}. \tag{1.11}$$

For the strong-type inequality with $p > 1$, Buckley **[Buc93]** proved the quantitative estimate

$$\|\mathcal{M}\|_{L^p(w)\longrightarrow L^p(w)} \le c_n p' [w]_{A_p}^{\frac{1}{p-1}}. \tag{1.12}$$

Moreover, the exponent $\frac{1}{p-1}$ is sharp, meaning that it cannot be replaced by any smaller power of $[w]_{A_p}$.

Using the Marcinkiewicz interpolation theorem (see **[Gra14, Theorem 1.3.2]** for example) with the trivial $L^\infty(w)$ strong estimate and the weak-type inequality (1.11), we obtain the following lemma.

Lemma 1.12. *Let $1 \le r < p < \infty$, and let $w \in A_r$. Then there exists a dimensional constant $c_n > 0$ such that*

$$\|\mathcal{M}f\|_{L^p(w)} \le c_n \left(\frac{p}{p-r}\right)^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{p}} \|f\|_{L^p(w)}. \tag{1.13}$$

In **[HP13]**, Hytönen and Pérez prove that the quantitative bound (1.12) can be improved in terms of a mixed $A_p - A_\infty$ constant.

Theorem 1.13. *Let $1 < p < \infty$, let $w \in A_p$, and denote $\sigma = w^{1-p'}$. Then there is a dimensional constant $c_n > 0$ such that*

$$\|\mathcal{M}\|_{L^p(w)\longrightarrow L^p(w)} \le c_n p' \left([w]_{A_p}[\sigma]_{A_\infty}\right)^{\frac{1}{p}}.$$

We recall the property that $[w]_{A_\infty} \le c_n [w]_{A_p}$ for all $p \ge 1$. Applying this inequality to the dual weight $\sigma \in A_{p'}$, we directly see that the mixed bound from Theorem 1.13 is an improvement of Buckley's estimate (1.12),

$$p' \left([w]_{A_p} [\sigma]_{A_\infty} \right)^{\frac{1}{p}} \le c_n p' \left([w]_{A_p} [\sigma]_{A_{p'}} \right)^{\frac{1}{p}} = c_n p' \left([w]_{A_p} [w]_{A_p}^{\frac{1}{p-1}} \right)^{\frac{1}{p}} = c_n p' [w]_{A_p}^{\frac{1}{p-1}}.$$

This demonstrates that the mixed bound is a strict improvement, as $[\sigma]_{A_\infty}$ can be much smaller than $[\sigma]_{A_{p'}}$.

Another fundamental property of the A_p classes is the reverse Hölder inequality, which in turn implies the left-openness property. We state the sharp reverse Hölder inequality for A_∞ weights in the Euclidean space obtained in **[HPR12, Theorem 2.3]**.

Theorem 1.14. *Let $w \in A_\infty$. Then for each $0 < \varepsilon \le \frac{1}{2^{n+1}[w]_{A_\infty} - 1}$, we have*

$$\frac{1}{|Q|} \int_Q w(x)^{1+\varepsilon} dx \le 2 \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{1+\varepsilon},$$

for every cube Q.

The range $0 < \varepsilon \le \frac{1}{2^{n+1}[w]_{A_\infty} - 1}$ is better than the one obtained in **[HP13]** and **[HPR12]** in the context of spaces of homogeneous type. This better range produces an improved precise open property. Since this specific quantitative bound is not explicitly proven in **[HPR12]**, we include the proof here.

Corollary 1.15. *Let $1 < p < \infty$ and let $w \in A_p$. We define the quantity*

$$\varepsilon = \frac{p-1}{\tau_n [\sigma]_{A_\infty}},$$

where $\tau_n = 2^{n+1}$ and $\sigma = w^{1-p'}$. Then, $w \in A_{p-\varepsilon}$ and

$$[w]_{A_{p-\varepsilon}} \le 2^{p-1} [w]_{A_p}.$$

Proof. Since $w \in A_p$, then $\sigma \in A_{p'} \subset A_\infty$. Let $r(\sigma) = 1 + \frac{1}{2^{n+1}[\sigma]_{A_\infty} - 1}$. Then, using Theorem 1.14, we have

$$\left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q \sigma(x)^{r(\sigma)} dx \right)^{\frac{p-1}{r(\sigma)}} \le \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{2}{|Q|} \int_Q \sigma(x) dx \right)^{p-1}.$$

We may choose $\varepsilon > 0$ such that $\frac{p-1}{r(\sigma)} = p - \varepsilon - 1$, namely,

$$\varepsilon = \frac{p-1}{r(\sigma)'} = \frac{p-1}{(1 + \frac{1}{2^{n+1}[\sigma]_{A_\infty} - 1})'} = \frac{p-1}{2^{n+1}[\sigma]_{A_\infty}}.$$

Note that $\varepsilon > 0$ and $p - \varepsilon > 1$. This yields that $w \in A_{p-\varepsilon}$. □

In **[PR18]**, a refined reverse Hölder inequality is established with an even better range for ε. While their original result provides a slightly larger admissible range, we state here an immediate consequence where this range is slightly restricted. By doing so, we obtain a much cleaner, explicit constant $2[w]_{A_\infty}$ on the right-hand side. A key point is that this restricted range remains sharp for flat weights, that is, when the Fujii–Wilson constant $[w]_{A_\infty}$ tends to 1.

Theorem 1.16. *Let $w \in A_\infty$. Then for each $0 \le \varepsilon \le \frac{1}{2^{n+1}([w]_{A_\infty}-1)}$,*

$$\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}\,dx \le 2[w]_{A_\infty}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)^{1+\varepsilon},$$

for every cube Q.

Let us mention that, in the case of A_1 weights, the first sharp reverse Hölder inequality was proved by Kinnunen **[Kin94]** in $\mathbb{R}^n$ with Lebesgue measure, and later extended to general measures in **[LPR17]**.

As an immediate consequence of Theorem 1.16, and using standard techniques, we obtain the following property concerning the weight measure of subsets.

Corollary 1.17. *Let $w \in A_\infty$. Then for each cube Q and each measurable subset $E \subset Q$, we have*

$$\frac{w(E)}{w(Q)} \le 2[w]_{A_\infty}\left(\frac{|E|}{|Q|}\right)^{\frac{1}{1+2^{n+1}([w]_{A_\infty}-1)}}.$$

Moreover, Theorem 1.16 yields another precise left-openness property. Compared with the result in **[HPR12]**, the value $p-\varepsilon$ that is obtained here is slightly smaller, but on the other hand, the constant is controlled by a mixed-type constant. Furthermore, this estimate captures the correct asymptotic behavior for flat weights, as we have $p-\varepsilon \to 1$ when $[\sigma]_{A_\infty} \to 1$.

Corollary 1.18. *Let $w \in A_p$ for some $1 < p < \infty$. If we consider*

$$\varepsilon = \frac{p-1}{2^{n+1}([\sigma]_{A_\infty}-1)+1}, \tag{1.14}$$

then $w \in A_{p-\varepsilon}$ with

$$[w]_{A_{p-\varepsilon}} \le (2[\sigma]_{A_\infty})^{p-1}[w]_{A_p}.$$

Remark 1.19. Observe that by raising the quantitative bound from Corollary 1.18 to the power of $1/p$, we obtain the following precise control:

$$[w]_{A_{p-\varepsilon}}^{\frac{1}{p}} \le 2^{\frac{1}{p'}}[\sigma]_{A_\infty}^{\frac{1}{p'}}[w]_{A_p}^{\frac{1}{p}} \le c_n[w]_{A_p}^{\frac{1}{p-1}}.$$

We next show that if a weight belongs to A_∞, then the associated measure $w\,dx$ is doubling, and we provide an explicit estimate of the doubling constant in terms of the Fujii–Wilson A_∞ constant of w. To the best of our knowledge, this type of quantitative result has not appeared in the literature before. Related estimates have been obtained using Hruščev's exponential A_∞ constant (see, for instance, **[Gra14]**).

Theorem 1.20. *Let $w \in A_\infty$, then the measure $w\,dx$ is doubling, i.e.,*

$$w(2Q) \le D_w w(Q),$$

for every cube Q, where

$$D_w \le \exp\left(\kappa\, n(4[w]_{A_\infty})^{1+2^{n+1}([w]_{A_\infty}-1)}\right),$$

and κ is a dimension-free constant.

Remark 1.21. Using the same argument together with the reverse Hölder inequality from **[HPR12]** produces the bound $D_w \lesssim \exp(\exp(c_n[w]_{A_\infty}))$.

Proof. Let Q be a cube and let $\lambda > 1$. Applying Corollary 1.17 to the cube λQ and the subset $E = \lambda Q \setminus Q$, we obtain

$$\begin{aligned}\frac{w(\lambda Q \setminus Q)}{w(\lambda Q)} \leq& 2[w]_{A_\infty}\left(\frac{|\lambda Q \setminus Q|}{|\lambda Q|}\right)^\theta\\ =&2[w]_{A_\infty}\left(1-\frac{|Q|}{|\lambda Q|}\right)^\theta\\ =&2[w]_{A_\infty}\left(1-\lambda^{-n}\right)^\theta,\end{aligned}$$

where $\theta = \frac{1}{1+2^{n+1}([w]_{A_\infty}-1)} \in (0,1)$. Hence,

$$\begin{aligned}w(\lambda Q) =& w(Q) + w(\lambda Q \setminus Q)\\ \leq& w(Q) + 2[w]_{A_\infty}\left(1-\lambda^{-n}\right)^\theta w(\lambda Q).\end{aligned}$$

We now choose $\lambda > 1$ such that

$$2[w]_{A_\infty}\left(1-\lambda^{-n}\right)^\theta \leq \frac{1}{2},$$

and consequently,

$$w(\lambda Q) \leq 2w(Q).$$

The previous inequality holds for any $1 < \lambda \leq \lambda_0$, where

$$\lambda_0 = \frac{1}{\left(1-\left(4[w]_{A_\infty}\right)^{-\frac{1}{\theta}}\right)^{\frac{1}{n}}}.$$

Let $m \in \mathbb{N}$ be the smallest integer such that $\lambda_0^m \geq 2$. Then, we have

$$w(2Q) \leq w(\lambda_0^m Q) \leq 2^m w(Q).$$

We observe that $m \simeq \frac{\log 2}{\log \lambda_0}$, so it remains to estimate $\log \lambda_0$. Since

$$\log \lambda_0 = -\frac{1}{n}\log(1-s),$$

where $s = (4[w]_{A_\infty})^{-\frac{1}{\theta}} \in (0, \frac{1}{4}]$, we can use the elementary inequality

$$s \leq -\log(1-s) \leq \frac{s}{1-s} \leq 2s,$$

which gives $\log \lambda_0 \simeq \frac{s}{n}$. Therefore,

$$D_w \leq 2^m \simeq 2^{\frac{\log 2}{\log \lambda_0}} \simeq \exp\left(\frac{\log 2}{\log_2 e}\frac{n}{s}\right) = \exp\left(\kappa\, n \left(4[w]_{A_\infty}\right)^{1+2^{n+1}([w]_{A_\infty}-1)}\right).$$

□

Remark 1.22. In the case of constant weights ($[w]_{A_\infty} = 1$), the doubling constant obtained above depends exponentially on the dimension, which is expected. Moreover, we observe that not only for constant weights, but also for flat weights of the form $[w]_{A_\infty} = 1 + \delta$ with $\delta \to 0$, we observe the correct asymptotic behavior $D_w \sim e^{cn}$.

We will also consider the class of pairs of A_p weights.

Definition 1.23. Let (u, v) be a pair of weights.

1. For $1 < p < \infty$ we say that $(u, v) \in A_p$ if
$$[u, v]_{A_p} = \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q u(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q v(x)^{1-p'}\, dx \right)^{p-1} < \infty.$$
2. We say that $(u, v) \in A_1$ if
$$[u, v]_{A_1} = \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q u(x)\, dx \right) \operatorname*{ess\,sup}_{x \in Q} \left(v(x)^{-1} \right) < \infty,$$
or equivalently, if $\mathcal{M}u(x) \leq [u, v]_{A_1} v(x)$ for a.e. $x \in \mathbb{R}^n$.

The relevance of the two-weight class A_p lies in the fact that it characterizes the weak-type boundedness of the maximal operator: $\mathcal{M}$ maps $L^p(v)$ into $L^{p,\infty}(u)$ if and only if $(u, v) \in A_p$. In contrast with the one-weight setting, this condition is, in general, not sufficient for strong-type boundedness from $L^p(v)$ to $L^p(u)$. The corresponding characterization was obtained by Sawyer **[Saw82]**.

Finally, we will use a weight class naturally adapted to Lorentz norms, introduced by Chung, Hunt, and Kurtz in **[CHK82]**.

Definition 1.24. Let $1 < p < \infty$ and let $1 \leq q < \infty$. We denote by $A_{p,q}$ the class of weights w such that
$$[w]_{A_{p,q}} = \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left\| \frac{1}{w} \right\|^p_{L^{p',q'}\left(Q, \frac{w(x)dx}{|Q|}\right)} < \infty.$$

Using (1.2), we can check that $w \in A_{p,q}$ implies the following weighted estimate analogous to (1.8): for every measurable $f \geq 0$ we have
$$\frac{1}{|Q|} \int_Q f(x)\, dx \leq [w]^{\frac{1}{p}}_{A_{p,q}} \|f\|_{L^{p,q}\left(Q, \frac{w(x)dx}{w(Q)}\right)}, \tag{1.15}$$
for every cube Q. In particular, for $1 < p < \infty$ and $1 < q < \infty$ we have $A_{p,q} = A_p$ (see **[CHK82, Theorem 4]**). At the endpoint $q = 1$, the class $A_{p,1}$ strictly contains A_p and is sometimes referred to in the literature as the restricted A_p class, denoted by $A_p^{\mathcal{R}}$.

In our weighted setting, we will use the following notation for the weighted Lorentz average over a cube Q:
$$\|f\|_{L^{p,q}\left(Q, \frac{w(x)dx}{w(Q)}\right)} = \left(p \int_0^\infty t^q \left(\frac{w(\{x \in Q : |f(x)| > t\})}{w(Q)} \right)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}},$$
whenever $q < \infty$, and
$$\|f\|_{L^{p,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} = \sup_{t>0} t \left(\frac{w(\{x \in Q : |f(x)| > t\})}{w(Q)} \right)^{\frac{1}{p}}.$$

1.4 Fractional operators

In this section, we introduce the fractional maximal operator and the Riesz potential. These operators are fundamental tools in harmonic analysis and play an important role in the study of Poincaré–Sobolev inequalities and related potential estimates.

1.4.1 Fractional maximal operator

Let $0 < \alpha < n$. The uncentered fractional maximal operator $\mathcal{M}_\alpha$ is defined for any locally integrable function f on $\mathbb{R}^n$ by

$$\mathcal{M}_\alpha f(x) = \sup_{Q \ni x} \frac{\ell(Q)^\alpha}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x. We observe that for $\alpha = 0$, this recovers the standard Hardy–Littlewood maximal function $\mathcal{M}$.

The mapping properties of $\mathcal{M}_\alpha$ are described in terms of exponents p and q satisfying

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

At the lower endpoint $p = 1$, the operator satisfies the weak-type estimate

$$\mathcal{M}_\alpha : L^1(\mathbb{R}^n) \longrightarrow L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n).$$

For $1 < p < \frac{n}{\alpha}$, it satisfies the strong-type bound

$$\mathcal{M}_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n).$$

These mapping properties can be extended to the weighted setting by utilizing the weight classes $A_{p,q}$ introduced by Muckenhoupt and Wheeden in **[MW74]**. It is important to emphasize that their fractional weight class is distinct from the Lorentz-adapted $A_{p,q}$ classes of Chung, Hunt, and Kurtz that we introduced in Section 1.3.2, despite the overlapping notation. Specifically, for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, the weighted strong-type inequality

$$\left(\int_{\mathbb{R}^n} (\mathcal{M}_\alpha f(x) w(x))^q \, dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} (|f(x)| w(x))^p \, dx \right)^{\frac{1}{p}}$$

holds if and only if w satisfies the Muckenhoupt–Wheeden $A_{p,q}$ condition. Similarly, the corresponding weak-type inequality at the endpoint $p = 1$ holds if and only if w satisfies their fractional $A_{1,q}$ condition. We refer also to **[LMPT10]** and the references therein for a modern approach.

1.4.2 Riesz potential

The Riesz potential of order α, with $0 < \alpha < n$, is defined as the convolution operator

$$I_\alpha f(x) = \gamma_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy,$$

where the normalizing constant $\gamma_{n,\alpha}$ is given in terms of the Gamma function by

$$\gamma_{n,\alpha} = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{\frac{n}{2}} 2^{\alpha} \Gamma\left(\frac{\alpha}{2}\right)}.$$

This precise constant is chosen so that I_α acts as a Fourier multiplier operator. Specifically, it ensures that the Fourier transform of the Riesz potential satisfies the relation

$$\widehat{I_\alpha f}(\xi) = (2\pi|\xi|)^{-\alpha} \widehat{f}(\xi),$$

where the Fourier transform is normalized as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx.$$

Unlike the maximal operator, which is well-defined for every locally integrable function, the integral defining $I_\alpha f(x)$ does not necessarily converge absolutely. A standard sufficient condition ensuring that $I_\alpha(|f|)(x) < \infty$ for almost every $x \in \mathbb{R}^n$ is that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y|)^{n-\alpha}} \, dy < \infty,$$

see for instance **[Lan72, p. 61]**.

The fractional maximal operator and the Riesz potential are closely related. Pointwise, the definition of the integral easily yields that the fractional maximal operator is controlled by the Riesz potential:

$$\mathcal{M}_\alpha f(x) \leq C_{n,\alpha} I_\alpha(|f|)(x) \tag{1.16}$$

for some constant $C_{n,\alpha} > 0$. The reverse inequality does not hold pointwise; however, the Riesz potential can be controlled by a combination of the maximal operator and the L^p norm of the function, thanks to a classical argument due to Hedberg **[Hed72]**.

Theorem 1.25. *Let $0 < \alpha < n$ and $1 \leq p < \frac{n}{\alpha}$. There exists a constant $C > 0$, depending only on n, α, and p, such that for every $f \in L^p(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$,*

$$|I_\alpha f(x)| \leq C \left(\mathcal{M} f(x)\right)^{1-\frac{\alpha p}{n}} \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\alpha p}{n}}.$$

Hedberg's inequality gives a direct and simple proof of the classical Hardy–Littlewood–Sobolev theorem for the Riesz potential. Specifically, for exponents p and q satisfying the relation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, we have the strong-type mapping

$$I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n),$$

whenever $1 < p < \frac{n}{\alpha}$. For the endpoint $p = 1$, the strong-type estimate fails, but the operator satisfies the weak-type mapping

$$I_\alpha : L^1(\mathbb{R}^n) \longrightarrow L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n).$$

It is a remarkable fact that the Riesz potential and the fractional maximal operator are comparable in $L^p(w)$ norm, even though they are not comparable pointwise. The non-trivial direction, controlling the norm of the Riesz potential by the norm of the fractional maximal operator, is quantified by the Muckenhoupt–Wheeden inequality, originally established in **[MW74]** using good-lambda methods.

Theorem 1.26. *Let $0 < \alpha < n$ and $0 < p < \infty$. If $w \in A_\infty$, then there exists a constant $C > 0$, depending only on n, α, and p, such that*

$$\|I_\alpha f\|_{L^p(w)} \leq C[w]_{A_\infty} \|\mathcal{M}_\alpha f\|_{L^p(w)}.$$

Remark 1.27. The explicit linear dependence on the Fujii–Wilson constant $[w]_{A_\infty}$ stated in Theorem 1.26 is not present in the original work of Muckenhoupt and Wheeden **[MW74]**. However, this precise quantitative bound can be deduced from the good-lambda inequality with exponential decay established by Honzík and Jaye **[HJ12]**, although it is not explicitly stated in their paper.

Combining this theorem with the trivial pointwise bound (1.16), it formally follows that the $L^p(w)$ norms of the Riesz potential and the fractional maximal operator are equivalent in norm whenever the underlying weight w belongs to the A_∞ class.

1.5 Poincaré-type inequalities

In this section, we establish Poincaré and Poincaré–Sobolev inequalities, which allow us to control the oscillation of a function in terms of its gradient. We first present pointwise subrepresentation formulas in terms of the Riesz potential to derive Poincaré–Sobolev inequalities. Finally, we introduce the truncation method to upgrade these estimates from weak norms to strong norms.

1.5.1 Subrepresentation formulas

A starting point in Sobolev theory is the following subrepresentation formula, which pointwise bounds the oscillation of a sufficiently smooth function by the first-order Riesz potential of its gradient.

Theorem 1.28. *Let $n \geq 2$. There exists a dimensional constant $c_n > 0$ such that for every $f \in C^1(\mathbb{R}^n)$,*

$$|f(x) - f_Q| \leq c_n I_1(|\nabla f| \chi_Q)(x),$$

for every $x \in Q$.

Proof. Fix $x \in Q$. By the triangle inequality and the Fundamental Theorem of Calculus along the segment joining x and y, we can bound the oscillation as

$$|f(x) - f_Q| \leq \frac{1}{|Q|} \int_Q |f(x) - f(y)| \, dy \leq \frac{1}{|Q|} \int_Q \int_0^1 |\nabla f(x + t(y - x))| \, |x - y| \, dt \, dy.$$

By Tonelli's theorem, we can interchange the order of integration. For each fixed $t \in (0, 1]$, we apply the affine change of variables $z = x + t(y - x)$ to the inner integral over Q. This transformation yields $dy = t^{-n} \, dz$ and $|x - y| = t^{-1}|x - z|$. Furthermore, the geometric constraint $y \in Q$ ensures that

$|x-y| \le \operatorname{diam}(Q)$, which gives us the lower bound $t \ge |x-z|/\operatorname{diam}(Q)$. Extending the upper limit of the t-integral to infinity to obtain a clean upper bound, we estimate

$$\begin{aligned}
|f(x) - f_Q| &\le \frac{1}{|Q|} \int_Q |\nabla f(z)|\, |x-z| \left(\int_{\frac{|x-z|}{\operatorname{diam}(Q)}}^{\infty} \frac{1}{t^{n+1}}\, dt \right) dz \\
&= \frac{\operatorname{diam}(Q)^n}{n|Q|} \int_Q \frac{|\nabla f(z)|}{|x-z|^{n-1}}\, dz \\
&= n^{n/2-1} \int_Q \frac{|\nabla f(z)|}{|x-z|^{n-1}}\, dz \\
&= c_n I_1(|\nabla f| \chi_Q)(x),
\end{aligned}$$

which completes the proof. □

Remark 1.29. If we additionally assume that f has compact support (or decays sufficiently fast at infinity), we can deduce a global representation formula. By taking an expanding sequence of cubes $\{Q_j\} \uparrow \mathbb{R}^n$ containing x, the averages f_{Q_j} tend to zero. Passing to the limit in Theorem 1.28, we obtain the global subrepresentation formula

$$|f(x)| \le c_n I_1(|\nabla f|)(x),$$

for every $x \in \mathbb{R}^n$.

This pointwise estimate provides a direct proof of the classical $(1,1)$-Poincaré inequality.

Proposition 1.30. *There exists a dimensional constant $c_n > 0$ such that for every $f \in C^1(\mathbb{R}^n)$, we have*

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx \le c_n \ell(Q) \frac{1}{|Q|} \int_Q |\nabla f(x)|\, dx,$$

for every cube Q.

Proof. Integrating the pointwise estimate from Theorem 1.28 over Q and using the self-adjointness of the Riesz potential (via Fubini's theorem), we obtain

$$\int_Q |f(x) - f_Q|\, dx \le c_n \int_Q |\nabla f(y)| I_1(\chi_Q)(y)\, dy.$$

For any $y \in Q$, we can uniformly bound $I_1(\chi_Q)(y)$ by enlarging the domain of integration to the bounding ball $B(y, \sqrt{n}\ell(Q))$:

$$I_1(\chi_Q)(y) \le \int_{B(y,\sqrt{n}\ell(Q))} \frac{1}{|y-z|^{n-1}}\, dz = c_n \ell(Q).$$

This concludes the proof. □

As we have just seen, the subrepresentation formula implies a local Poincaré inequality. This implication can be reversed: assuming a local Poincaré-type condition on cubes is sufficient to deduce a subrepresentation formula.

We will need the following generalized version of this principle. The result can be found in **[FLW96]** for the case $r = 1$ and in **[FH00]** for $r > 1$; however, we will use the precise quantitative version from **[HMPV23]**, which carefully tracks the dependence on the parameters.

Lemma 1.31. *Let $Q_0 \subset \mathbb{R}^n$ be a cube. Assume that $0 < \alpha < n$ and $1 \le r < \infty$. Suppose that there exist a constant $\kappa > 0$, a function $f \in L^1(Q_0)$, and a non-negative measurable function g such that*

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le \kappa \ell(Q)^\alpha \left(\frac{1}{|Q|}\int_Q g(x)^r\,dx\right)^{\frac{1}{r}},$$

for every cube $Q \subset Q_0$. Then, there exists a dimensional constant $C_n > 0$ such that

$$|f(x) - f_{Q_0}| \le C_n \frac{\kappa}{\gamma_{n,\alpha}^{\frac{1}{r}} \alpha^{\frac{1}{r'}} (n-\alpha)^{\frac{1}{r}}} \ell(Q_0)^{\frac{\alpha}{r'}} \left(I_\alpha(g^r \chi_{Q_0})(x)\right)^{\frac{1}{r}},$$

for every Lebesgue point $x \in Q_0$ of f.

Remark 1.32. By setting $r = 1$, $\alpha = 1$, and $g = |\nabla f|$, the conjugate exponent is $r' = \infty$, which makes the geometric factor $\ell(Q_0)^{\frac{1}{r'}} = 1$. In this case, Lemma 1.31 shows that the local $(1,1)$-Poincaré inequality implies the subrepresentation formula in Theorem 1.28. Combined with the converse implication proved above, this shows that the local $(1,1)$-Poincaré inequality and the subrepresentation formula are equivalent, up to dimensional constants.

Proof. Let $E \subset Q_0$ be the complement of the set of Lebesgue points of f, so that $|E| = 0$. Fix $x \in Q_0 \setminus E$. We choose a nested chain $\{Q_k\}_{k=1}^\infty \subset \mathcal{D}(Q_0)$ of dyadic subcubes such that $Q_1 = Q_0$, $Q_{k+1} \subset Q_k$, $|Q_{k+1}| = 2^{-n}|Q_k|$, $x \in Q_k$ for every k, and $\bigcap_{k\ge1} Q_k = \{x\}$.

Since x is a Lebesgue point, $f_{Q_k} \to f(x)$ as $k \to \infty$. Telescoping the sequence of averages we have,

$$\begin{aligned}
|f(x) - f_{Q_0}| &\le \sum_{k=1}^\infty |f_{Q_{k+1}} - f_{Q_k}| \\
&\le \sum_{k=1}^\infty \frac{2^n}{|Q_k|}\int_{Q_k} |f(y) - f_{Q_k}|\,dy \\
&\le 2^n \kappa \sum_{k=1}^\infty \ell(Q_k)^\alpha \left(\frac{1}{|Q_k|}\int_{Q_k} g(y)^r\,dy\right)^{\frac{1}{r}}.
\end{aligned}$$

Assume first $1 < r < \infty$. Factoring the term outside the integral as $\left(\ell(Q_k)^\alpha\right)^{\frac{1}{r'}}\left(\ell(Q_k)^{\alpha-n}\right)^{\frac{1}{r}}$, we apply the Hölder inequality to obtain

$$|f(x) - f_{Q_0}| \le 2^n \kappa \left(\sum_{k=1}^\infty \ell(Q_k)^\alpha\right)^{\frac{1}{r'}} \left(\sum_{k=1}^\infty \ell(Q_k)^{\alpha-n}\int_{Q_k} g(y)^r\,dy\right)^{\frac{1}{r}}.$$

The first series is a standard geometric sum since $\ell(Q_k) = 2^{-(k-1)}\ell(Q_0)$. Using the bound $1 - 2^{-\alpha} \ge c_n\alpha$, we get

$$\left(\sum_{k=1}^\infty \ell(Q_k)^\alpha\right)^{\frac{1}{r'}} \le \left(\frac{\ell(Q_0)^\alpha}{1-2^{-\alpha}}\right)^{\frac{1}{r'}} \le c_n \frac{\ell(Q_0)^{\frac{\alpha}{r'}}}{\alpha^{\frac{1}{r'}}}.$$

For the second series, we use Fubini's theorem to interchange the sum and the integral

$$\sum_{k=1}^\infty \ell(Q_k)^{\alpha-n}\int_{Q_0} g(y)^r \chi_{Q_k}(y)\,dy = \int_{Q_0} g(y)^r \left(\sum_{k=1}^\infty \ell(Q_k)^{\alpha-n}\chi_{Q_k}(y)\right)dy.$$

Fix $y \in Q_0 \setminus \{x\}$. The indicator function $\chi_{Q_k}(y)$ is non-zero only if both $x, y \in Q_k$, which implies $|x-y| \le \sqrt{n}\ell(Q_k)$. Thus, the sum is restricted to indices $k \le k_0(y)$, where $k_0(y)$ is the largest integer such that $\ell(Q_k) \ge |x-y|/\sqrt{n}$. Since $\alpha < n$, the series is geometric with ratio $2^{n-\alpha} > 1$, meaning it is dominated by its final term. Using that $1 - 2^{-(n-\alpha)} \ge c_n(n-\alpha)$ and the definition of $k_0(y)$, we bound the inner sum pointwise by

$$\sum_{k=1}^{k_0(y)} \ell(Q_k)^{\alpha-n} \le \frac{\ell(Q_{k_0(y)})^{\alpha-n}}{1-2^{-(n-\alpha)}} \le \frac{C_n}{n-\alpha}|x-y|^{\alpha-n}.$$

Inserting these bounds back into our initial Hölder estimate yields

$$\begin{aligned} |f(x) - f_{Q_0}| &\le C_n \frac{\kappa}{\alpha^{\frac{1}{r'}}} \ell(Q_0)^{\frac{\alpha}{r'}} \left(\int_{Q_0} g(y)^r \frac{|x-y|^{\alpha-n}}{n-\alpha}\, dy \right)^{\frac{1}{r}} \\ &= C_n \frac{\kappa}{\gamma_{n,\alpha}^{\frac{1}{r}} \alpha^{\frac{1}{r'}} (n-\alpha)^{\frac{1}{r}}} \ell(Q_0)^{\frac{\alpha}{r'}} \left(I_\alpha(g^r \chi_{Q_0})(x) \right)^{\frac{1}{r}}. \end{aligned}$$

Finally, in the endpoint case $r = 1$, the exact same argument applies by simply omitting the Hölder step. The first block $\ell(Q_0)^{\alpha/r'}$ equals 1, the parameter α vanishes from the denominator since $1/r' = 0$, and the telescoping sum directly yields the fractional integral bound, completing the proof. □

1.5.2 Poincaré–Sobolev inequalities

We say that a function f satisfies a local $(1,p)$-Poincaré inequality on cubes, where $1 \le p < \infty$, if there exists a constant $c_n > 0$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\, dx \le c_n \ell(Q) \left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\, dx \right)^{\frac{1}{p}},$$

for every cube Q. A remarkable feature of this inequality is its self-improving property: for any $1 \le p < \infty$, the local $(1,p)$-Poincaré inequality implies the stronger (p,p)-Poincaré inequality. That is, there exists a constant $C > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p\, dx \right)^{\frac{1}{p}} \le C\ell(Q) \left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\, dx \right)^{\frac{1}{p}},$$

for every cube Q. The first self-improving results of this type go back to Saloff-Coste **[Sal92]**; see also Hajłasz and Koskela **[HK00]** for the extension to metric spaces with a doubling measure. This phenomenon will be studied in more detail in Chapter 2. Furthermore, when the L^1 (or L^p) average on the left-hand side is replaced by an L^q average with $q > p$, namely,

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^q\, dx \right)^{\frac{1}{q}} \le C\ell(Q) \left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\, dx \right)^{\frac{1}{p}},$$

for every cube Q, we say that the function satisfies a Poincaré–Sobolev inequality. If $n \ge 2$ and $1 \le p < n$, the optimal exponent q for which this integrability gain holds is the classical Sobolev exponent p^*, originally defined in **[Sob63]**. This critical exponent is defined by the relation

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}, \tag{1.17}$$

which yields the explicit expression

$$p^* = \frac{np}{n-p}.$$

Using the subrepresentation formula stated in Theorem 1.28 and the mapping properties of the Riesz potential, we can directly inherit the integrability gains of the fractional integral by taking the appropriate norms on both sides. If $1 < p < n$, applying the strong-type Hardy–Littlewood–Sobolev bound for I_1 yields the Poincaré–Sobolev inequality. That is, there exists a constant $c_{n,p} > 0$ such that for every cube Q, we have

$$\left(\int_Q |f(x) - f_Q|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq c_{n,p} \left(\int_Q |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

We now make two remarks concerning this estimate. First, the integral on the right-hand side is restricted to the cube Q precisely because the gradient in the subrepresentation formula is multiplied by the characteristic function χ_Q. Second, we can normalize this inequality by dividing both sides by their corresponding powers of the measure of Q. Using the relation (1.17) and the fact that $|Q|^{\frac{1}{n}} = \ell(Q)$, we obtain

$$\left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq c_{n,p} \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

However, this fractional integral approach does not allow us to establish the strong-type estimate at the lower endpoint $p = 1$, which is nevertheless known to be true in the classical Sobolev setting. Instead, by relying on the weak-type mapping properties of I_1, we can deduce a weak-type Poincaré–Sobolev inequality that is valid for the full range $1 \leq p < n$. Namely, there exists a constant $c_{n,p} > 0$ such that the following weak-type estimate holds

$$\|f - f_Q\|_{L^{p^*,\infty}\left(Q, \frac{dx}{|Q|}\right)} \leq c_{n,p} \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}},$$

for every cube Q.

As we have seen, the fractional integral approach yields a weak-type Poincaré–Sobolev inequality. A remarkable feature of having a first-order gradient on the right-hand side is that it allows the estimate to self-improve from a weak-type norm to a stronger Lorentz norm. The truncation method is a classical technique for upgrading these weak-type Poincaré–Sobolev inequalities to stronger results along the Lorentz scale. In the first-order setting, this idea already appears in the early work of Maz'ya **[Maz85, p. 110]** (see also **[Haj01]** for a detailed discussion).

First, we will need the following lemma.

Lemma 1.33. *Let μ be a measure on a set X such that $\mu(X) < \infty$. Let $v : X \to [0, \infty)$ be a measurable function satisfying*

$$\mu(\{x \in X : v(x) = 0\}) \geq \frac{\mu(X)}{2}.$$

Then, for every $t > 0$,

$$\mu(\{x \in X : v(x) > t\}) \leq 2 \inf_{c \in \mathbb{R}} \mu\left(\left\{ x \in X : |v(x) - c| > \frac{t}{2} \right\} \right).$$

Proof. Fix $t > 0$ and $c \in \mathbb{R}$. If $c \leq t/2$, then $\{v > t\} \subseteq \{|v - c| > t/2\}$, and the estimate is immediate. If $c > t/2$, then for every $x \in \{v = 0\}$ we have $|v(x) - c| = |c| = c > \frac{t}{2}$, and hence $\{v = 0\} \subseteq \{|v - c| > t/2\}$. Therefore,

$$\mu\left(\left\{|v - c| > \frac{t}{2}\right\}\right) \geq \mu(\{v = 0\}) \geq \frac{\mu(X)}{2}.$$

Since $\{v > t\} \subseteq X$, it follows that

$$\mu(\{v > t\}) \leq \mu(X) \leq 2\mu\left(\left\{|v - c| > \frac{t}{2}\right\}\right).$$

Thus the estimate holds for every $c \in \mathbb{R}$. Taking the infimum over c completes the proof. □

Given a function v and $0 < t_1 < t_2 < \infty$, we define the truncated function

$$v_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1, & \text{if } v(x) \geq t_2, \\ v(x) - t_1, & \text{if } t_1 < v(x) < t_2, \\ 0, & \text{if } v(x) \leq t_1. \end{cases}$$

We will use two variants of the truncation argument: one for gradients in Lorentz spaces, which we state now, and one for fractional seminorms that will be introduced in the next section.

Theorem 1.34. *Let $1 < q < \infty$ and $1 \leq p \leq q$, let $Q \subset \mathbb{R}^n$ be a cube, and let μ and ν be Borel measures on $\mathbb{R}^n$, with $\mu(Q) = 1$. Assume $1 \leq r \leq p$. If there exists a constant $C > 0$ such that for every $f \in \mathrm{Lip}(Q)$,*

$$\left\|f - f_{Q,\mu}\right\|_{L^{q,\infty}(Q,\mu)} \leq C \left\|\nabla f\right\|_{L^{p,r}(Q,\nu)},$$

then there exists $c_q > 0$ such that for every $f \in \mathrm{Lip}(Q)$ we have

$$\left\|f - f_{Q,\mu}\right\|_{L^{q,p}(Q,\mu)} \leq c_q C \left\|\nabla f\right\|_{L^{p,r}(Q,\nu)}.$$

The proof of this theorem follows the general strategy outlined in **[KO03]**. The main idea is that by truncating the function at different height levels, the gradients of the resulting truncated functions are supported on pairwise disjoint sets. This disjointness of the supports is the key ingredient that allows us to sum the localized estimates. In this step, we use the summability property of Lorentz quasi-norms (formally established in Lemma 1.3), and this is also the point where the restriction $1 \leq r \leq p$ is required.

Proof. Fix $f \in \mathrm{Lip}(Q)$. Let $b \in \mathbb{R}$ be a median of f over Q with respect to μ, that is

$$\mu(\{x \in Q : f(x) \geq b\}) \geq \frac{\mu(Q)}{2} \quad \text{and} \quad \mu(\{x \in Q : f(x) \leq b\}) \geq \frac{\mu(Q)}{2}.$$

We split $f - b$ into its positive and negative parts, setting $v_+ = \max\{f - b, 0\}$ and $v_- = -\min\{f - b, 0\}$. In the sequel, let v denote either v_+ or v_-. Without loss of generality, we may assume that

$0 \le v(x) < \infty$ for all $x \in Q$. Let $0 < t_1 < t_2 < \infty$. Observe that, due to our choice of the median, $\mu(\{x \in Q : v_{t_1}^{t_2}(x) = 0\}) \ge \frac{\mu(Q)}{2}$. Applying Lemma 1.33 to $v_{t_1}^{t_2}$, we get, for every $t > 0$,

$$\begin{aligned} t\,\mu\left(\{x \in Q : v_{t_1}^{t_2}(x) > t\}\right)^{\frac{1}{q}} &\le 2^{\frac{1}{q}+1} \inf_{a\in\mathbb{R}} \left\|v_{t_1}^{t_2} - a\right\|_{L^{q,\infty}(Q,\mu)} \\ &\le 2^{\frac{1}{q}+1} \left\|v_{t_1}^{t_2} - (v_{t_1}^{t_2})_{Q,\mu}\right\|_{L^{q,\infty}(Q,\mu)} \\ &\le 2^{\frac{1}{q}+1} C \left\|\nabla v_{t_1}^{t_2}\right\|_{L^{p,r}(Q,\nu)}, \end{aligned}$$

where we invoked the hypothesis of the theorem, valid since truncations of Lipschitz functions remain Lipschitz. Evaluating this estimate at dyadic levels $t = 2^{k-1}$, $t_1 = 2^k$ and $t_2 = 2^{k+1}$, and observing that $|\nabla v_{2^k}^{2^{k+1}}| = |\nabla v|\chi_{A_k}$ almost everywhere on the sets $A_k = \{x \in Q : 2^k < v(x) \le 2^{k+1}\}$, we obtain

$$\sum_{k\in\mathbb{Z}} 2^{kp}\,\mu\left(\{x \in Q : v_{2^k}^{2^{k+1}}(x) > 2^{k-1}\}\right)^{\frac{p}{q}} \le 2^{\frac{p}{q}+2p} C^p \sum_{k\in\mathbb{Z}} \left\||\nabla v|\chi_{A_k}\right\|_{L^{p,r}(Q,\nu)}^p.$$

Since the level sets A_k are pairwise disjoint, applying Lemma 1.3, we have

$$\sum_{k\in\mathbb{Z}} 2^{kp}\,\mu\left(\{x \in Q : v_{2^k}^{2^{k+1}}(x) > 2^{k-1}\}\right)^{\frac{p}{q}} \le 2^{\frac{p}{q}+2p} C^p \left\|\nabla v\right\|_{L^{p,r}(Q,\nu)}^p,$$

for $1 \le r \le p$. To relate this discrete sum to the actual Lorentz norm of v, we use the inclusion $\{x \in Q : v(x) > 2^k\} \subseteq \{x \in Q : v_{2^{k-1}}^{2^k}(x) > 2^{k-2}\}$:

$$\begin{aligned} \|v\|_{L^{q,p}(Q,\mu)}^p &\le p \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} t^{p-1} \mu(\{x \in Q : v(x) > t\})^{\frac{p}{q}}\, dt \\ &\le \sum_{k\in\mathbb{Z}} \left((2^{k+1})^p - (2^k)^p\right) \mu(\{x \in Q : v(x) > 2^k\})^{\frac{p}{q}} \\ &\le (2^p - 1)2^p \sum_{k\in\mathbb{Z}} 2^{kp} \mu(\{x \in Q : v_{2^k}^{2^{k+1}}(x) > 2^{k-1}\})^{\frac{p}{q}} \\ &\le (2^p - 1)2^{3p+\frac{p}{q}} C^p \left\|\nabla v\right\|_{L^{p,r}(Q,\nu)}^p. \end{aligned}$$

We have proved the estimate for both v_+ and v_-. Since $f - b = v_+ - v_-$ and $|\nabla v_\pm| \le |\nabla f|$ almost everywhere, we have

$$\begin{aligned} \inf_{c\in\mathbb{R}} \|f - c\|_{L^{q,p}(Q,\mu)} &\le \|f - b\|_{L^{q,p}(Q,\mu)} \\ &\le \left(\|v_+\|_{L^{q,p}(Q,\mu)} + \|v_-\|_{L^{q,p}(Q,\mu)}\right) \\ &\le 2^{4+\frac{1}{q}} C \left(\|\nabla v_+\|_{L^{p,r}(Q,\nu)} + \|\nabla v_-\|_{L^{p,r}(Q,\nu)}\right) \\ &\le 64C \|\nabla f\|_{L^{p,r}(Q,\nu)}. \end{aligned}$$

Finally, by Proposition 1.6, applied with $p = 1$, together with (1.4), we get

$$\begin{aligned} \left\|f - f_{Q,\mu}\right\|_{L^{q,p}(Q,\mu)} &\le 2q' \inf_{c\in\mathbb{R}} \|f - c\|_{L^{q,p}(Q,\mu)} \\ &\le 128q'C \|\nabla f\|_{L^{p,r}(Q,\nu)}. \end{aligned}$$

This completes the proof. □

We will also use a non-standard version of this argument, based on a trick from **[FPW98]**, which allows us to keep the unweighted average f_Q on the left-hand side. In order to do this, besides the weak-type Poincaré–Sobolev inequality, we require an additional unweighted L^1 Poincaré inequality.

Theorem 1.35. *Let $1 \le p \le q < \infty$, $1 \le r \le p$, let $Q \subset \mathbb{R}^n$ be a cube, and let μ and ν be Borel measures on $\mathbb{R}^n$ such that $\mu(Q) = 1$. Assume that there exists a constant $C > 0$ such that for every $f \in \mathrm{Lip}(Q)$,*

$$\left\|f - f_Q\right\|_{L^{q,\infty}(Q,\mu)} \le C \left\|\nabla f\right\|_{L^{p,r}(Q,\nu)}$$

and

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le C \left\|\nabla f\right\|_{L^{p,r}(Q,\nu)}.$$

Then, for every $f \in \mathrm{Lip}(Q)$, we have

$$\left\|f - f_Q\right\|_{L^{q,p}(Q,\mu)} \le 10C \left\|\nabla f\right\|_{L^{p,r}(Q,\nu)}.$$

The previous theorem can be proved by following the strategy in **[FPW98, Theorem 3.1]** and using the summability property of Lorentz quasi-norms stated in Lemma 1.3 above, this is also the point where the restriction $1 \le r \le p$ enters.

Proof. Fix $f \in \mathrm{Lip}(Q)$ and set $v := |f - f_Q|$ and $\lambda := \frac{1}{|Q|}\int_Q v(x)\,dx$. If $\lambda = 0$, then $v = 0$ almost everywhere with respect to Lebesgue measure. Since v is Lipschitz, it follows that $v = 0$ in Q, and the result is trivial. Thus, we may assume that $\lambda > 0$. For $k \ge 0$, let us define $\lambda_k := 2^k\lambda$,

$$E_k := \{x \in Q : v(x) > \lambda_k\},$$

and, for $k \ge 1$,

$$A_k := E_{k-1} \setminus E_k = \{x \in Q : \lambda_{k-1} < v(x) \le \lambda_k\}.$$

By the definition of the Lorentz quasi-norm and the fact that $\mu(Q) = 1$, we have

$$\begin{aligned}
\|v\|^p_{L^{q,p}(Q,\mu)} &= p\int_0^\infty t^p \mu(\{x \in Q : v(x) > t\})^{\frac{p}{q}}\,\frac{dt}{t} \\
&\le p\int_0^{2\lambda} t^p\,\frac{dt}{t} + p\sum_{k=1}^\infty \int_{\lambda_k}^{\lambda_{k+1}} t^p \mu(E_k)^{\frac{p}{q}}\,\frac{dt}{t} \\
&\le (2\lambda)^p + \sum_{k=2}^\infty \lambda_k^p \mu(E_{k-1})^{\frac{p}{q}} \\
&\le 2^p C^p \|\nabla f\|^p_{L^{p,r}(Q,\nu)} + \sum_{k=2}^\infty \lambda_k^p \mu(E_{k-1})^{\frac{p}{q}},
\end{aligned}$$

where in the last inequality we use the L^1 Poincaré inequality. We now estimate the terms in the sum. Fix $k \ge 2$. If $x \in E_{k-1}$, then $v(x) > \lambda_{k-1} = 2\lambda_{k-2}$, and therefore $v^{\lambda_{k-1}}_{\lambda_{k-2}}(x) = \lambda_{k-2}$. On the other hand, since $2v^{2t}_t \le v$ for every $t > 0$, we have

$$(v^{\lambda_{k-1}}_{\lambda_{k-2}})_Q = \frac{1}{|Q|}\int_Q v^{\lambda_{k-1}}_{\lambda_{k-2}}(x)\,dx \le \frac{1}{2|Q|}\int_Q v(x)\,dx = \frac{\lambda}{2} \le \frac{\lambda_{k-2}}{2}.$$

Hence, for every $x \in E_{k-1}$,

$$\begin{aligned}\lambda_{k-2} &= v_{\lambda_{k-2}}^{\lambda_{k-1}}(x)\\ &\le \left|v_{\lambda_{k-2}}^{\lambda_{k-1}}(x) - (v_{\lambda_{k-2}}^{\lambda_{k-1}})_Q\right| + (v_{\lambda_{k-2}}^{\lambda_{k-1}})_Q\\ &\le \left|v_{\lambda_{k-2}}^{\lambda_{k-1}}(x) - (v_{\lambda_{k-2}}^{\lambda_{k-1}})_Q\right| + \frac{\lambda_{k-2}}{2}.\end{aligned}$$

Therefore,

$$E_{k-1} \subseteq \left\{x \in Q : \left|v_{\lambda_{k-2}}^{\lambda_{k-1}}(x) - (v_{\lambda_{k-2}}^{\lambda_{k-1}})_Q\right| \ge \frac{\lambda_{k-2}}{2}\right\}.$$

Applying the weak-type assumption to the Lipschitz function $v_{\lambda_{k-2}}^{\lambda_{k-1}}$, we obtain

$$\begin{aligned}\lambda_k \mu(E_{k-1})^{\frac{1}{q}} &\le 8\frac{\lambda_{k-2}}{2}\mu\left(\left\{\left|v_{\lambda_{k-2}}^{\lambda_{k-1}} - (v_{\lambda_{k-2}}^{\lambda_{k-1}})_Q\right| \ge \frac{\lambda_{k-2}}{2}\right\}\right)^{\frac{1}{q}}\\ &\le 8C\|\nabla v_{\lambda_{k-2}}^{\lambda_{k-1}}\|_{L^{p,r}(Q,\nu)}.\end{aligned}$$

Hence,

$$\sum_{k=2}^{\infty} \lambda_k^p \mu(E_{k-1})^{\frac{p}{q}} \le 8^p C^p \sum_{k=2}^{\infty} \|\nabla v_{\lambda_{k-2}}^{\lambda_{k-1}}\|_{L^{p,r}(Q,\nu)}^p.$$

Finally, for $k \ge 2$ we have $|\nabla v_{\lambda_{k-2}}^{\lambda_{k-1}}| = |\nabla v|\chi_{A_{k-1}}$ almost everywhere in Q. Since the sets A_k are pairwise disjoint, Lemma 1.3 yields

$$\begin{aligned}\sum_{k=2}^{\infty} \|\nabla v_{\lambda_{k-2}}^{\lambda_{k-1}}\|_{L^{p,r}(Q,\nu)}^p &= \sum_{k=2}^{\infty} \|\nabla v\, \chi_{A_{k-1}}\|_{L^{p,r}(Q,\nu)}^p\\ &\le \|\nabla v\|_{L^{p,r}(Q,\nu)}^p\\ &\le \|\nabla f\|_{L^{p,r}(Q,\nu)}^p,\end{aligned}$$

where we use $1 \le r \le p$. Combining the previous estimates, we conclude that

$$\begin{aligned}\|f - f_Q\|_{L^{q,p}(Q,\mu)}^p &= \|v\|_{L^{q,p}(Q,\mu)}^p\\ &\le 2^p C^p \|\nabla f\|_{L^{p,r}(Q,\nu)}^p + 8^p C^p \|\nabla f\|_{L^{p,r}(Q,\nu)}^p\\ &\le 10^p C^p \|\nabla f\|_{L^{p,r}(Q,\nu)}^p.\end{aligned}$$

Taking p-th roots completes the proof. □

As an immediate application of the truncation method, we can strictly improve the weak-type Poincaré–Sobolev inequality established via the fractional integral approach. This refinement in the Lorentz scale was first observed by O'Neil **[ONe63]** and Peetre **[Pee66a]**. By setting $q = p^*$ and $r = p$ in Theorem 1.34, the normalized weak-type estimate upgrades to the $L^{p^*,p}$ Lorentz norm.

Theorem 1.36. *Let $n \ge 2$ and let $1 \le p < n$. There exists a constant $c_{n,p} > 0$ such that for every $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, we have*

$$\|f - f_Q\|_{L^{p^*,p}\left(Q,\frac{dx}{|Q|}\right)} \le c_{n,p}\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,dx\right)^{\frac{1}{p}},$$

for every cube Q.

Remark 1.37. In the Lorentz scale, the target space $L^{p^*,p}$ is optimal and cannot be replaced by $L^{p^*,s}$ for any $s < p$ (see **[EKP00]**). In fact, $L^{p^*,p}$ is the smallest possible target space for the Sobolev embedding of $W^{1,p}$, not only within the Lorentz scale but even among all rearrangement-invariant spaces (e.g. Orlicz spaces, Lebesgue spaces, etc.). Furthermore, for a detailed study concerning the optimal constant $c_{n,p}$ in this inequality, we refer to **[CRT13]** and **[Iok14]**.

1.5.3 Local and global inequalities

Up to this point, our analysis has focused on local inequalities formulated on cubes. A natural question is how these local estimates relate to their global counterparts on the whole space $\mathbb{R}^n$, and subsequently, to more general domains.

The passage from local inequalities on cubes to global inequalities on $\mathbb{R}^n$ can be made rigorous by a limiting argument on functions with sufficient decay. Consider a compactly supported smooth function $f \in C_c^\infty(\mathbb{R}^n)$. We can choose an expanding sequence of cubes $\{Q_j\} \uparrow \mathbb{R}^n$. Since f is compactly supported, its integral over Q_j becomes constant for large j, and therefore the average $f_{Q_j} \to 0$ as the measure $|Q_j| \to \infty$. Thus,

$$|f(x)|^{p^*} = \liminf_{j\to\infty} |f(x) - f_{Q_j}|^{p^*} \chi_{Q_j}(x),$$

for every $x \in \mathbb{R}^n$. By applying Fatou's Lemma and the local Poincaré–Sobolev inequality on each cube Q_j, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |f(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}} &\le \liminf_{j\to\infty} \left(\int_{Q_j} |f(x) - f_{Q_j}|^{p^*}\,dx\right)^{\frac{1}{p^*}} \\ &\le c_{n,p} \liminf_{j\to\infty} \left(\int_{Q_j} |\nabla f(x)|^p\,dx\right)^{\frac{1}{p}} \\ &\le c_{n,p} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx\right)^{\frac{1}{p}}.\end{aligned}$$

In this way, the local estimate implies the classical global Sobolev inequality. Namely, for $1 \le p < n$, there exists a constant $C_{n,p} > 0$ such that

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \le C_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)},$$

for every $f \in C_c^\infty(\mathbb{R}^n)$, and by standard density arguments, for every $f \in W^{1,p}(\mathbb{R}^n)$. The exact same limiting procedure applies to Theorem 1.36, yielding the sharp global estimate

$$\|f\|_{L^{p^*,p}(\mathbb{R}^n)} \le C_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

Conversely, once the global inequality is established, we can deduce local inequalities for a wide class of sets beyond cubes by utilizing the Sobolev extension theory. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently regular boundary (such as a Lipschitz domain). Classical extension theory provides a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ satisfying $Ef = f$ almost

everywhere in Ω. In particular, the gradient of the extended function is controlled by the full Sobolev norm of the original function,

$$\|\nabla(Ef)\|_{L^p(\mathbb{R}^n)} \le C_E \left(\|\nabla f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

By evaluating the global Sobolev inequality on the extended function Ef and restricting the integration domain on the left-hand side to Ω, we obtain

$$\|f\|_{L^{p^*}(\Omega)} \le \|Ef\|_{L^{p^*}(\mathbb{R}^n)} \le C\|\nabla(Ef)\|_{L^p(\mathbb{R}^n)} \le C\left(\|\nabla f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

To recover the Poincaré–Sobolev inequality, which bounds the oscillation of the function, we apply this estimate to the zero-mean function $g = f - f_\Omega$. Since $|\nabla g| = |\nabla f|$, this yields

$$\|f - f_\Omega\|_{L^{p^*}(\Omega)} \le C\left(\|\nabla f\|_{L^p(\Omega)} + \|f - f_\Omega\|_{L^p(\Omega)}\right).$$

Finally, to bound the L^p norm of the oscillation on the right-hand side, we use the standard (p, p)-Poincaré inequality valid on Lipschitz domains, namely $\|f - f_\Omega\|_{L^p(\Omega)} \le C\|\nabla f\|_{L^p(\Omega)}$. This directly absorbs the L^p norm into the gradient term, yielding the desired local Poincaré–Sobolev estimate

$$\|f - f_\Omega\|_{L^{p^*}(\Omega)} \le C\|\nabla f\|_{L^p(\Omega)}.$$

1.5.4 Fractional Poincaré-type inequalities

In the previous section, we bounded the oscillation of a function by the L^p norm of its gradient on the right-hand side. A natural generalization is to study the fractional Sobolev seminorm, which integrates finite differences over the domain.

Given $p \ge 1$ and a fractional parameter $0 < \delta < 1$, we can readily establish a basic fractional analogue of the $(1, p)$-Poincaré inequality. Namely, there exists a constant $c_n > 0$ such that for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and every cube Q,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le c_n \ell(Q)^\delta \left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}. \tag{1.18}$$

This estimate follows directly from Jensen's inequality and the geometric bound $|x - y| \le \sqrt{n}\ell(Q)$ for points in Q:

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx &\le \frac{1}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|\,dy\,dx \\
&\le \left(\frac{1}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|^p\,dy\,dx\right)^{\frac{1}{p}} \\
&\le \left(\frac{(\sqrt{n}\ell(Q))^{n+\delta p}}{|Q|^2}\int_Q\int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}} \\
&= c_n \ell(Q)^\delta \left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}.
\end{aligned}$$

However, inequality (1.18) is not sharp in its dependence on the parameter δ. In **[BBM02]**, Bourgain, Brezis, and Mironescu proved a refined version of this inequality that includes a key extra gain

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le c_n(1-\delta)^{\frac{1}{p}}\ell(Q)^\delta\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}. \tag{1.19}$$

The appearance of the factor $(1-\delta)^{\frac{1}{p}}$ is not merely a technical artifact, it reflects the correct asymptotic behavior of the fractional seminorm as $\delta \to 1^-$. In a seminal result, Bourgain, Brezis, and Mironescu **[BBM01]** demonstrated that this exact seminorm recovers the classical local gradient in the limit for every $f \in W^{1,p}(Q)$,

$$\lim_{\delta\to 1^-}(1-\delta)^{\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}} = C_1\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

In fact, the factor $(1-\delta)^{\frac{1}{p}}$ is strictly necessary, without it, the limit is infinite for any non-constant function **[Bre02, Corollary 1]**. The extra gain factor $(1-\delta)^{\frac{1}{p}}$ has recently been called the BBM phenomenon **[HMPV23]**.

Moreover, it was also proved in **[BBM01]** that the right-hand side of (1.19) satisfies the uniform bound

$$(1-\delta)^{\frac{1}{p}}\ell(Q)^\delta\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}} \le c_n\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,dx\right)^{\frac{1}{p}}. \tag{1.20}$$

At the other endpoint of the fractional scale, Maz'ya and Shaposhnikova **[MS02; MS03]** showed that introducing a factor of $\delta^{\frac{1}{p}}$ allows the global fractional seminorm to recover the L^p norm of the function itself as $\delta \to 0^+$. Specifically, there exists a constant $C_2 = C_2(n,p)$ such that

$$\lim_{\delta\to 0^+}\delta^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}} = C_2\left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

These fractional estimates also satisfy a self-improving property. For any $p \ge 1$, the L^1 oscillation on the left-hand side of (1.19) can be upgraded to an L^p norm. Namely, there exists a constant $c_{n,p} > 0$ such that for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p\,dx\right)^{\frac{1}{p}} \le c_{n,p}(1-\delta)^{\frac{1}{p}}\ell(Q)^\delta\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}.$$

Furthermore, in the range $1 \le p < \frac{n}{\delta}$, this self-improving technique shows that (1.19) actually implies the following weak fractional Poincaré–Sobolev inequality,

$$\|f-f_Q\|_{L^{p^*_\delta,\infty}\left(Q,\frac{dx}{|Q|}\right)} \le c_{n,p}(1-\delta)^{\frac{1}{p}}\ell(Q)^\delta\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}, \tag{1.21}$$

where the fractional Sobolev exponent p^*_δ is defined by the relation

$$\frac{1}{p}-\frac{1}{p^*_\delta} = \frac{\delta}{n},$$

which yields the explicit expression

$$p^*_\delta := \frac{np}{n-\delta p}.$$

Furthermore, we can also reach the optimal Lorentz space $L^{p^*_\delta,p}$, mirroring the sharpest results for the gradient, stated in Theorem 1.36. As shown in **[DLV23]**, the truncation method is also applicable to fractional seminorms. We will rely on the following version, which merges the ideas from **[DLV23]** with the weak-implies-strong argument of **[KO03]** to improve it up to the Lorentz norm $L^{q,p}$.

Theorem 1.38. *Let $1 < q < \infty$ and $1 \le p \le q$, let μ be a Borel measure on $\mathbb{R}^n$ with $\mu(Q) = 1$, let $Q \subset \mathbb{R}^n$ be a cube, and let $K : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be a nonnegative measurable function. Assume that there exists a constant $C > 0$ such that for every $f \in \mathrm{Lip}(Q)$,*

$$\left\|f - f_{Q,\mu}\right\|_{L^{q,\infty}(Q,\mu)} \le C\left(\int_Q\int_Q |f(y)-f(z)|^p K(y,z)\,dy\,dz\right)^{\frac{1}{p}}.$$

Then there exists a constant $c_{p,q} > 0$ such that for every $f \in \mathrm{Lip}(Q)$,

$$\left\|f - f_{Q,\mu}\right\|_{L^{q,p}(Q,\mu)} \le c_{p,q}C\left(\int_Q\int_Q |f(y)-f(z)|^p K(y,z)\,dy\,dz\right)^{\frac{1}{p}}.$$

Proof. Fix $f \in \mathrm{Lip}(Q)$. Let $b \in \mathbb{R}$ be a median of f over Q with respect to μ, that is

$$\mu(\{x \in Q :\ f(x) \ge b\}) \ge \frac{\mu(Q)}{2} \quad \text{and} \quad \mu(\{x \in Q :\ f(x) \le b\}) \ge \frac{\mu(Q)}{2}.$$

We split $f-b$ into its positive and negative parts, setting $v_+ = \max\{f-b,0\}$ and $v_- = -\min\{f-b,0\}$. In the sequel, let v denote either v_+ or v_-. Since f is Lipschitz on Q, so are v_+ and v_-, and in particular $0 \le v(x) < \infty$ for every $x \in Q$. Let $0 < t_1 < t_2 < \infty$. Observe that, due to our choice of the median, $\mu(\{x \in Q : v^{t_2}_{t_1}(x) = 0\}) \ge \frac{\mu(Q)}{2}$. Applying Lemma 1.33 to $v^{t_2}_{t_1}$, we get, for every $t > 0$,

$$\begin{aligned} t\,\mu(\{x \in Q : v^{t_2}_{t_1}(x) > t\})^{\frac{1}{q}} &\le 2^{\frac{1}{q}+1} \inf_{c\in\mathbb{R}} \left\|v^{t_2}_{t_1} - c\right\|_{L^{q,\infty}(Q,\mu)} \\ &\le 2^{\frac{1}{q}+1} \left\|v^{t_2}_{t_1} - (v^{t_2}_{t_1})_{Q,\mu}\right\|_{L^{q,\infty}(Q,\mu)} \\ &\le 2^{\frac{1}{q}+1} C\left(\int_Q\int_Q |v^{t_2}_{t_1}(y) - v^{t_2}_{t_1}(z)|^p K(y,z)\,dy\,dz\right)^{\frac{1}{p}}, \end{aligned}$$

where we invoked the hypothesis of the theorem, valid since truncations of Lipschitz functions remain Lipschitz. Hence,

$$t^p \mu(\{x \in Q : v^{t_2}_{t_1}(x) > t\})^{\frac{p}{q}} \le c_{p,q}C^p \int_Q\int_Q |v^{t_2}_{t_1}(y) - v^{t_2}_{t_1}(z)|^p K(y,z)\,dy\,dz. \tag{1.22}$$

We now estimate the $L^{q,p}(Q,\mu)$ quasi-norm of v,

$$\begin{aligned} \|v\|^p_{L^{q,p}(Q,\mu)} &= p\int_0^\infty t^{p-1}\mu(\{x \in Q : v(x) > t\})^{\frac{p}{q}}\,dt \\ &\le \sum_{k\in\mathbb{Z}} \mu(\{x \in Q : v(x) > 2^{k-1}\})^{\frac{p}{q}} \int_{2^{k-1}}^{2^k} pt^{p-1}\,dt \\ &= (2^p-1)\sum_{k\in\mathbb{Z}} 2^{kp}\mu(\{x \in Q : v(x) > 2^k\})^{\frac{p}{q}}. \end{aligned}$$

Since $\{x \in Q : v(x) > 2^k\} \subset \left\{x \in Q : v_{2^{k-1}}^{2^k}(x) > 2^{k-2}\right\}$, we can apply (1.22) with $t = 2^{k-2}$, $t_1 = 2^{k-1}$ and $t_2 = 2^k$. Hence

$$\begin{aligned} 2^{kp}\mu(\{x \in Q : v(x) > 2^k\})^{\frac{p}{q}} &\leq c_p 2^{(k-2)p}\mu\left(\left\{x \in Q : v_{2^{k-1}}^{2^k}(x) > 2^{k-2}\right\}\right)^{\frac{p}{q}} \\ &\leq c_{p,q}C^p \int_Q \int_Q |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y,z)\, dy\, dz. \end{aligned}$$

Summing over $k \in \mathbb{Z}$, we get

$$\|v\|_{L^{q,p}(Q,\mu)}^p \leq c_{p,q}C^p \sum_{k\in\mathbb{Z}} \int_Q \int_Q |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y,z)\, dy\, dz. \tag{1.23}$$

It remains to control the sum on the right-hand side. For $k \in \mathbb{Z}$, set

$$E_k = \{x \in Q :\ v(x) > 2^k\}, \qquad A_k = E_{k-1} \setminus E_k,$$

and also

$$A_{-\infty} = \{x \in Q :\ v(x) = 0\}.$$

Since v is finite, we have the disjoint decomposition

$$Q = A_{-\infty} \cup \bigcup_{i\in\mathbb{Z}} A_i.$$

For each $k \in \mathbb{Z}$, the truncation $v_{2^{k-1}}^{2^k}$ is constant on the regions where $v \leq 2^{k-1}$ and where $v \geq 2^k$. Therefore, using the preceding dyadic decomposition, we obtain

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \int_Q \int_Q |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y,z)\, dy\, dz \\ &\quad\leq \sum_{k\in\mathbb{Z}} \sum_{-\infty\leq i\leq k} \sum_{j\geq k} \int_{A_i} \int_{A_j} |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y,z)\, dy\, dz \\ &\qquad+ \sum_{k\in\mathbb{Z}} \sum_{i\geq k} \sum_{-\infty\leq j\leq k} \int_{A_i} \int_{A_j} |v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p K(y,z)\, dy\, dz. \end{aligned} \tag{1.24}$$

We estimate the first term. Let $y \in A_i$ and $z \in A_j$, with $-\infty \leq i \leq k \leq j$. If $j - 1 > i$, then

$$|v(y) - v(z)| \geq |v(z)| - |v(y)| \geq 2^{j-1} - 2^i \geq 2^{j-2},$$

and,

$$|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)| \leq 2^{k-1} \leq 2^k.$$

Consequently,

$$|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)| \leq 4 \cdot 2^{k-j}|v(y) - v(z)|.$$

On the other hand, the estimate

$$|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)| \leq |v(y) - v(z)|.$$

holds for every $k \in \mathbb{Z}$. Therefore,

$$|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)| \leq 4 \cdot 2^{k-j}|v(y) - v(z)|, \tag{1.25}$$

whenever $-\infty \le i \le k \le j$ and $(y,z) \in A_i \times A_j$. Using (1.25), we get

$$\sum_{k\in\mathbb{Z}} \sum_{-\infty\le i\le k} \sum_{j\ge k} \int_{A_i}\int_{A_j} |v^{2^k}_{2^{k-1}}(y) - v^{2^k}_{2^{k-1}}(z)|^p K(y,z)\,dy\,dz$$
$$\le 4^p \sum_{k\in\mathbb{Z}} \sum_{-\infty\le i\le k} \sum_{j\ge k} 2^{p(k-j)} \int_{A_i}\int_{A_j} |v(y)-v(z)|^p K(y,z)\,dy\,dz.$$

Changing the order of summation and using $\sum_{k=i}^{j} 2^{p(k-j)} \le \sum_{\ell=0}^{\infty} 2^{-p\ell} = \frac{1}{1-2^{-p}}$, we obtain

$$\sum_{k\in\mathbb{Z}} \sum_{-\infty\le i\le k} \sum_{j\ge k} \int_{A_i}\int_{A_j} |v^{2^k}_{2^{k-1}}(y) - v^{2^k}_{2^{k-1}}(z)|^p K(y,z)\,dy\,dz$$
$$\le \frac{4^p}{1-2^{-p}} \sum_{-\infty\le i\le j} \int_{A_i}\int_{A_j} |v(y)-v(z)|^p K(y,z)\,dy\,dz$$
$$\le \frac{4^p}{1-2^{-p}} \int_Q\int_Q |v(y)-v(z)|^p K(y,z)\,dy\,dz.$$

The second term in the right-hand side of (1.24) is estimated in the same way, with the roles of the variables interchanged. Combining the two estimates, we conclude that

$$\sum_{k\in\mathbb{Z}} \int_Q\int_Q |v^{2^k}_{2^{k-1}}(y) - v^{2^k}_{2^{k-1}}(z)|^p K(y,z)\,dy\,dz \le c_p \int_Q\int_Q |v(y)-v(z)|^p K(y,z)\,dy\,dz, \tag{1.26}$$

where $c_p = 2\frac{4^p}{1-2^{-p}}$. Combining (1.23) with (1.26), we obtain

$$\|v\|^p_{L^{q,p}(Q,\mu)} \le c_{p,q} C^p \int_Q\int_Q |v(y)-v(z)|^p K(y,z)\,dy\,dz. \tag{1.27}$$

We now apply (1.27) to v_+ and v_-. Since

$$|v_\pm(y) - v_\pm(z)| \le |f(y)-f(z)|, \qquad y,z\in Q,$$

and since $|f-b| = v_+ + v_-$ with disjoint supports, we obtain

$$\|f-b\|^p_{L^{q,p}(Q,\mu)} \le \|v_+\|^p_{L^{q,p}(Q,\mu)} + \|v_-\|^p_{L^{q,p}(Q,\mu)}$$
$$\le c_{p,q} C^p \int_Q\int_Q |f(y)-f(z)|^p K(y,z)\,dy\,dz.$$

Finally, since $\mu(Q) = 1$ and $q > 1$, by (1.4) we have

$$|f_{Q,\mu} - b| \le \int_Q |f-b|\,d\mu \le c_{p,q}\|f-b\|_{L^{q,p}(Q,\mu)}.$$

Using the quasi-triangle inequality and the fact that $\|\chi_Q\|_{L^{q,p}(Q,\mu)} = 1$, we conclude

$$\|f - f_{Q,\mu}\|_{L^{q,p}(Q,\mu)} \le c_{p,q}\left(\|f-b\|_{L^{q,p}(Q,\mu)} + |f_{Q,\mu}-b|\right)$$
$$\le c_{p,q}\|f-b\|_{L^{q,p}(Q,\mu)}$$
$$\le c_{p,q} C \left(\int_Q\int_Q |f(y)-f(z)|^p K(y,z)\,dy\,dz\right)^{\frac{1}{p}}.$$

□

We also use the following version using ideas of **[FPW98, Theorem 3.1]** in order to keep the unweighted average f_Q on the left-hand side. We refer to **[LW26, Proposition A.2]** for an explicit proof of a related result.

Theorem 1.39. *Let $1 \le p \le q < \infty$, let $Q \subset \mathbb{R}^n$ be a cube, let μ be a Borel measure on $\mathbb{R}^n$ such that $\mu(Q) = 1$, and let $K : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be a nonnegative measurable function. Assume that there exists $C > 0$ such that for every $f \in \mathrm{Lip}(Q)$,*

$$\left\|f - f_Q\right\|_{L^{q,\infty}(Q,\mu)} \le C \left(\int_Q \int_Q |f(y) - f(z)|^p K(y,z)\, dy\, dz \right)^{\frac{1}{p}},$$

and

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx \le C \left(\int_Q \int_Q |f(y) - f(z)|^p K(y,z)\, dy\, dz \right)^{\frac{1}{p}}.$$

Then, for every $f \in \mathrm{Lip}(Q)$, we have

$$\left\|f - f_Q\right\|_{L^{q,p}(Q,\mu)} \le 130C \left(\int_Q \int_Q |f(y) - f(z)|^p K(y,z)\, dy\, dz \right)^{\frac{1}{p}}.$$

Proof. Following the definitions and notation introduced in the proof of Theorem 1.35, we obtain

$$\begin{aligned}
\|v\|^p_{L^{q,p}(Q,\mu)} &= p \int_0^\infty t^p \mu(\{x \in Q : v(x) > t\})^{\frac{p}{q}} \frac{dt}{t} \\
&\le p \int_0^{2\lambda} t^p \frac{dt}{t} + p \sum_{k=1}^\infty \int_{\lambda_k}^{\lambda_{k+1}} t^p \mu(E_k)^{\frac{p}{q}} \frac{dt}{t} \\
&\le (2\lambda)^p + \sum_{k=1}^\infty \lambda_{k+1}^p \mu(E_k)^{\frac{p}{q}} \\
&\le 2^p C^p \int_Q \int_Q |f(y) - f(z)|^p K(y,z)\, dy\, dz + \sum_{k=1}^\infty \lambda_{k+1}^p \mu(E_k)^{\frac{p}{q}}.
\end{aligned}$$

On the other hand, exactly as in the proof of Theorem 1.35, we have

$$E_k \subset \left\{ x \in Q : \left| v^{\lambda_k}_{\lambda_{k-1}}(x) - (v^{\lambda_k}_{\lambda_{k-1}})_Q \right| \ge \frac{\lambda_{k-1}}{2} \right\}, \qquad k \ge 1.$$

Therefore, applying the weak-type assumption to $v^{\lambda_k}_{\lambda_{k-1}}$, we obtain

$$\sum_{k=1}^\infty \lambda_{k+1}^p \mu(E_k)^{\frac{p}{q}} \le 8^p C^p \sum_{k=1}^\infty \int_Q \int_Q |v^{\lambda_k}_{\lambda_{k-1}}(y) - v^{\lambda_k}_{\lambda_{k-1}}(z)|^p K(y,z)\, dy\, dz.$$

Finally, the remaining sum on the right-hand side is estimated exactly as in the proof of Theorem 1.38. More precisely, applying (1.26), we obtain

$$\sum_{k=1}^\infty \int_Q \int_Q |v^{\lambda_k}_{\lambda_{k-1}}(y) - v^{\lambda_k}_{\lambda_{k-1}}(z)|^p K(y,z)\, dy\, dz \le c_p \int_Q \int_Q |f(y) - f(z)|^p K(y,z)\, dy\, dz,$$

where we observe that $c_p \le 16^p$. Combining the previous estimates, we conclude that

$$\|f - f_Q\|_{L^{q,p}(Q,\mu)} \le 130C \left(\int_Q \int_Q |f(y) - f(z)|^p K(y,z)\, dy\, dz \right)^{\frac{1}{p}}.$$

□

Combining the weak-type fractional Poincaré–Sobolev inequality (1.21) with the fractional truncation method, we obtain the following result.

Theorem 1.40. *Let $0 < \delta < 1$ and let $1 \le p < \frac{n}{\delta}$. Then there exists a constant $c_{n,p} > 0$ such that, for every cube $Q \subset \mathbb{R}^n$ and every $f \in W^{\delta,p}(Q)$, we have*

$$\|f - f_Q\|_{L^{p^*_\delta,p}\left(Q,\frac{dx}{|Q|}\right)} \le c_{n,p}(1-\delta)^{\frac{1}{p}}\ell(Q)^{\delta}\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}.$$

Chapter 2

Self-improving generalized Poincaré inequalities to Poincaré–Sobolev inequalities and applications

In this chapter, we present the results of the following work, with some detailed proofs.

[CPZ26] Claros, A., Pérez, C. and Zheng, L. *From Generalized Poincaré to Poincaré-Sobolev inequalities via self-improving methods*, arXiv preprint arXiv:2606.08556 (2026).

The results developed here study the self-improving phenomena for generalized Poincaré-type inequalities mentioned in Chapter 1. More precisely, we show how a discrete summability condition on the functional controlling the local oscillation leads to quantitative generalized Poincaré–Sobolev estimates. The later sections apply this principle to weighted first-order and fractional Poincaré–Sobolev inequalities, Lorentz refinements of the gradient term, polynomial and vector-valued oscillations, and extensions to rectangles.

2.1 Introduction

The classical local $(1,1)$ Poincaré inequality asserts that there exists a dimensional constant $c_n > 0$ such that for every $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ we have

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le c_n\,\ell(Q)\,\frac{1}{|Q|}\int_Q |\nabla f(x)|\,dx, \tag{2.1}$$

for every cube $Q \subset \mathbb{R}^n$.

This inequality enjoys a self-improving property that has been the focus of recent studies. Notably, this phenomenon does not stem from the presence of the gradient on the right-hand side of (2.1), but

rather is a consequence of a discrete summation condition inherent to the functional therein. Motivated by this, we consider inequalities of the form

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le a(Q), \tag{2.2}$$

valid for every cube Q, where a is a functional defined on the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Inequalities of this type, featuring a local oscillation term on the left-hand side and a functional $a(Q)$ on the right, will be referred to as generalized Poincaré inequalities.

Generalized Poincaré inequalities and their self-improving properties were first studied in a general framework by Franchi, Pérez, and Wheeden **[FPW98]** (see also **[MP98; LP05]** for further refinements). In this setting, the functional a is required to satisfy a suitable geometric discrete summation condition. This approach provides a way to obtain weighted Poincaré–Sobolev inequalities without relying on classical representation formulas. For a recent overview of this approach and its connections with harmonic analysis, including Poincaré–Sobolev, Trudinger, and John–Nirenberg type inequalities, we refer to **[Pér26]**.

A major motivation for studying weighted Poincaré–Sobolev inequalities comes from the regularity theory of degenerate elliptic equations. In their seminal work, Fabes, Kenig, and Serapioni **[FKS82]** used such inequalities as part of the Moser iteration scheme to prove local Hölder regularity for weak solutions of degenerate elliptic equations whose ellipticity is controlled by an A_2 weight (see also **[HKM06]**).

While the self-improving results in these earlier works hold in the more general setting of spaces of homogeneous type, the specific geometry of cubes in $\mathbb{R}^n$ (with sides parallel to the axes) allows for a more delicate analysis. In the present chapter, we exploit this framework to obtain more precise estimates than those treated in **[FPW98]** or **[MP98]**. Within this context, the discrete summation condition introduced in **[FPW98]** can be stated as follows.

Definition 2.1. Let w be any weight in $\mathbb{R}^n$. We say that the functional a satisfies the weighted $D_p(w)$ condition for $0 < p < \infty$ if there is a constant C such that, for any cube Q and any family $\{Q_j\}_j$ of pairwise disjoint subcubes of Q, the following inequality holds:

$$\sum_j a(Q_j)^p w(Q_j) \le C^p a(Q)^p w(Q).$$

The best possible constant C above is denoted by $\|a\|_{D_p(w)}$, and in this case we write $a \in D_p(w)$.

In **[FPW98]** it was proved that if $w \in A_\infty$ and $a \in D_p(w)$, then the starting point inequality (2.2) implies a weighted weak L^p generalized Poincaré inequality. More recently, the dependence of the constants in the main result was improved in **[CP21]**, in the context of cubes as a consequence of an extension of the John–Nirenberg theorem. For completeness, we state the latest version of this result below.

Theorem 2.2. *Let $w \in A_\infty$ and let $1 < p < \infty$. Consider a functional a satisfying the $D_p(w)$ condition with constant $\|a\|_{D_p(w)}$. Let f be a locally integrable function such that*

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le a(Q),$$

for every cube Q. Then, for every cube Q,

$$\left\|f - f_Q\right\|_{L^{p,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq c_n p [w]_{A_\infty} \|a\|_{D_p(w)}\, a(Q). \tag{2.3}$$

As a consequence of the so-called Kolmogorov inequality, (2.3) yields the following strong estimate:

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^q w(x)\,dx\right)^{\frac{1}{q}} \leq c_n \left(\frac{p}{p-q}\right)^{\frac{1}{q}} p [w]_{A_\infty} \|a\|_{D_p(w)}\; a(Q),$$

for all $0 < q < p$. Thus, the $D_p(w)$ condition implies that (2.2) improves to a weighted L^q inequality for every $0 < q < p$, although in general it is not known whether it yields the strong norm L^p bound.

In the case of functionals for which the truncation method (or a weak-implies-strong argument) is applicable, (2.3) does indeed imply the L^p estimate (see **[KO03]**). More recently, the condition introduced in Definition 2.1 was sharpened in **[PR19]**. This more refined formulation is satisfied by the fundamental examples and was specifically designed to yield direct strong-type L^p estimates, in contrast to the weak-type L^p bounds obtained under the original D_p condition. The new condition, denoted by $SD_p^s(w)$, is stronger than $D_p(w)$, yet it is satisfied by a broad class of functionals (see **[PR19; CMPR23; HMPV23]** for examples in a variety of settings).

The precise definition is the following.

Definition 2.3. Let w be any weight in $\mathbb{R}^n$, and let $s > 0$. We say that the functional a satisfies the weighted $SD_p^s(w)$ condition for $0 < p < \infty$ if there is a constant C such that, for any cube Q and any family $\{Q_j\}$ of pairwise disjoint subcubes of Q, the following inequality holds:

$$\sum_j a(Q_j)^p w(Q_j) \leq C^p \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q).$$

The best possible constant C above is denoted by $\|a\|_{SD_p^s(w)}$, and in this case we write $a \in SD_p^s(w)$.

The constants $\|a\|_{D_p(w)}$ and $\|a\|_{SD_p^s(w)}$ are always greater than or equal to 1; we shall omit the subscript when there is no ambiguity; moreover, when working with Lebesgue measure (i.e. $w = 1$) we will also drop the dependence on w and simply write $\|a\|_{D_p}$ and $\|a\|_{SD_p^s}$. This condition may be viewed as a strengthened version of the $D_p(w)$ condition, incorporating an additional smallness preservation factor on the right-hand side. A typical example of a functional satisfying this condition is

$$a(Q) = \ell(Q)^\alpha \left(\frac{\nu(Q)}{w(Q)}\right)^{\frac{1}{p}},$$

where ν is a locally finite measure and $\alpha, p > 0$. It was shown in **[PR19, Lemma 3.2]** that $a \in SD_p^{n/\alpha}(w)$.

The main self-improving result of **[PR19]** states that if $w \in A_\infty$ and $a \in SD_p^s(w)$, then the inequality (2.2) implies a weighted L^p generalized Poincaré inequality. In that paper, Pérez and Rela conjectured that the A_∞ assumption could be removed. A partial step in this direction was later obtained by Martínez-Perales **[Mar21]**. More recently, the conjecture was fully resolved by Lerner, Lorist, and Ombrosi **[LLO22]**. Their approach relies on sparse domination to derive a pointwise estimate and,

moreover, yields an improved dependence on s in the resulting constant. One may ask if the assumption $w \in A_\infty$ in the D_p-type self-improving result, Theorem 2.2, can also be removed; the answer is no. We construct a counterexample in Proposition 2.10 to illustrate this phenomenon.

We now state the self-improving result proved in **[LLO22]**.

Theorem 2.4. *Let $1 < p < \infty$. Consider a functional a satisfying the $SD_p^s(w)$ condition with $s > 0$ and constant $\|a\|$. Let f be a locally integrable function such that*

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le a(Q),$$

for every cube Q. Then there exists a dimensional constant c_n such that, for any cube Q,

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^p w(x)dx\right)^{\frac{1}{p}} \le c_n(1+s)\,\|a\|\,a(Q). \tag{2.4}$$

The $SD_p^s(w)$ condition is notably flexible and has found applications in several contexts, including the multi-parameter setting of rectangles **[CMPR23]** and fractional Poincaré–Sobolev inequalities **[HMPV23]**.

2.2 Main self-improving result

The purpose of this section is to establish the main self-improving theorem of the chapter. We work under the assumption that the functional a satisfies an $SD_p^s(w)$ condition and that the weight belongs to A_r for some $1 \le r < \infty$. The resulting gain depends on the "dimensional" parameter rs: below this threshold we obtain a weak-type Sobolev improvement, while in the borderline and supercritical regime the conclusion is of exponential type. We shall also show later that the exponent obtained in the weak-type range is sharp for a wide range of values of r.

Theorem 2.5. *Let $1 \le p < \infty$, $1 \le r < \infty$, $1 \le s < \infty$, and let a be a functional over cubes. Let $w \in A_r$ and let f be a locally integrable function satisfying*

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le a(Q),$$

for every cube Q.

- *Assume that $a \in SD_p^s(w)$ and $p < rs$. Then there exists a dimensional constant $c_n > 0$ such that, for every cube Q,*

$$\left\|f - f_Q\right\|_{L^{p^*_{r,s},\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le c_n p^*_{r,s}\,[w]_{A_\infty}\,[w]_{A_r}^{\frac{1}{rs}}\,\|a\|\,a(Q), \tag{2.5}$$

where $p^*_{r,s}$ is defined by the relation

$$\frac{1}{p} - \frac{1}{p^*_{r,s}} = \frac{1}{r\,s}.$$

- *Assume that $a \in SD_p^s(w)$ and $p \geq rs$. Then there exists a constant $c_{n,p} > 0$ such that, for every cube Q,*

$$\|f - f_Q\|_{\exp L\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \leq c_{n,p}\,[w]_{A_\infty}\,[w]_{A_r}^{\frac{1}{p}}\,\|a\|\,a(Q). \tag{2.6}$$

Here $\|\cdot\|_{\exp L(X,\mu)}$ denotes the Luxemburg norm associated with the Young function $\Phi(t) = e^t - 1$, namely

$$\|f\|_{\exp L(X,\mu)} := \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

Estimate (2.5) reflects the compatibility between the D_p and the SD_p^s conditions. Indeed, by Hölder's inequality, if $1 \leq p < s$ and

$$p_s^* = \frac{sp}{s-p}$$

then

$$D_{p_s^*} \subseteq SD_p^s \subseteq SD_1^{(p_s^*)'},$$

with

$$\|a\|_{SD_1^{(p_s^*)'}} \leq \|a\|_{SD_p^s} \leq \|a\|_{D_{p_s^*}}.$$

Thus, in the case $r = 1$, Theorem 2.5 shows that the SD_p^s assumption already yields the sharp weak-$L^{p_s^*}$ generalized Poincaré inequality associated with the $D_{p_s^*}$ condition. In this sense, (2.5) improves the estimates obtained in **[PR19; LLO22]** when w is an A_p weight.

Estimate (2.6) extends **[PR19, Corollary 1.9]** in two directions: no monotonicity assumption on the functional a is required, and the argument works in the weighted setting under the more general condition $a \in SD_p^s(w)$.

Remark 2.6. Consider the unweighted setting $w = 1$. Then $w \in A_1$, and therefore we may take $r = 1$ in Theorem 2.5. In the classical Poincaré inequality,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \leq c_n \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,dx\right)^{\frac{1}{p}},$$

the corresponding functional belongs to SD_p^n. Thus $s = n$ is the natural dimensional parameter in this model case. If $1 \leq p < n$, the exponent $p_{1,n}^*$ in (2.5) is precisely the classical Sobolev exponent $p^* = \frac{np}{n-p}$. On the other hand, in the borderline and supercritical regime $p \geq n$, equivalently $p \geq rs$ with $r = 1$ and $s = n$, Theorem 2.5 yields the exponential endpoint (2.6), in agreement with the classical Trudinger-type behavior.

Remark 2.7. The exponent $p_{r,s}^*$ in (2.5) is sharp in the range $1 \leq r \leq p$ and $s \geq n$. Indeed, in Propositions 2.11 and 2.12 we construct weights $w \in A_r$ and smooth test functions for which the conclusion of (2.5) fails if $p_{r,s}^*$ is replaced by any larger exponent.

In Section 2.7, we extend Theorem 2.5 to higher-order oscillations by replacing the average f_Q, in both the hypothesis and the conclusion, with an appropriate polynomial $P_Q^m f$. This polynomial denotes the projection of f onto the space of polynomials of degree m in n variables over Q, leading to corollaries of Poincaré-type inequalities involving higher-order derivatives. In Section 2.8, we extend

it to vector-valued generalized Poincaré inequalities. Specifically, we consider functions $f : \mathbb{R}^n \longrightarrow \ell_q$, where the oscillation in both the hypothesis and the conclusion is measured with respect to the ℓ_q-norm, appearing as $\big\|f - f_Q\big\|_{\ell_q}$. Furthermore, in Section 2.9, it is also extended to the case of rectangles in $\mathbb{R}^n$.

Before proving Theorem 2.5, we record a simple extrapolation lemma from weak-L^p estimates with linear growth in p to exponential integrability. This will be used only in the endpoint and supercritical regime $p \geq rs$, where the weak-type conclusion obtained from the subcritical argument is converted into the exponential estimate (2.6). The following proposition is a refinement of **[CP21, Proposition 2.3]**.

Proposition 2.8. *Let (X, μ) be a probability space and let $f \geq 0$ be measurable. Assume that there exist constants $\gamma > 0$ and $p_0 \geq 1$ such that for every $p > p_0$,*

$$\|f\|_{L^{p,\infty}(X,\mu)} \leq \gamma\, p. \tag{2.7}$$

Then, $f \in \exp L(X, \mu)$, in the sense that

$$\mu(\{x \in X : \ f(x) > t\}) \leq e^{p_0} \exp\Big(-\frac{t}{e\,\gamma}\Big),$$

for every $t > 0$. Moreover, there exists an absolute constant $C > 0$ such that

$$\|f\|_{\exp L(X,\mu)} \leq C\, p_0\, \gamma.$$

Proof. Fix $t > e\gamma p_0$. Choose $p = p(t) := \frac{t}{e\gamma} > p_0$, so that $\gamma p/t = e^{-1}$. Applying (2.7) with this choice of p gives

$$\mu(\{x \in X : f(x) > t\}) \leq \Big(\frac{1}{e}\Big)^{p(t)} = \exp\Big(-\frac{t}{e\gamma}\Big),$$

for every $t > e\gamma p_0$. If $0 < t \leq e\gamma p_0$, then $\mu(\{x \in X : f(x) > t\}) \leq 1$ and hence

$$\mu(\{x \in X : f(x) > t\}) \leq e^{p_0} \exp\Big(-\frac{t}{e\gamma}\Big),$$

since $\exp(-t/(e\gamma)) \geq e^{-p_0}$ in the range $0 < t \leq e\gamma p_0$. Therefore, for every $t > 0$,

$$\mu(\{x \in X : f(x) > t\}) \leq e^{p_0} \exp\Big(-\frac{t}{e\gamma}\Big).$$

□

We shall also use the following local good-lambda type inequality from **[CP21, Corollary 1.4]**. The version below, formulated in terms of the localized dyadic sharp maximal function, follows from **[CP21, Remark 3.1]**.

Proposition 2.9. *Let f be a locally integrable function and let $w \in A_\infty$. Then there exist dimensional constants $c_1, c_2 > 0$ such that*

$$w\left(\Big\{x \in Q : \mathcal{M}_Q(f - f_Q)(x) > \lambda,\ \mathcal{M}_Q^{\#} f(x) \leq \gamma\lambda\Big\}\right) \leq c_1 \exp\left(-\frac{c_2}{\gamma [w]_{A_\infty}}\right) w(Q),$$

for every $\lambda, \gamma > 0$ and every cube Q, where

$$\mathcal{M}_Q^{\#} f(x) := \sup_{\substack{x \in R \\ R \in \mathcal{D}(Q)}} \frac{1}{|R|} \int_R |f(y) - f_R|\, dy.$$

With Propositions 2.8 and 2.9 in hand, we now turn to the proof of Theorem 2.5.

Proof of Theorem 2.5. We first show that estimate (2.6) follows directly from estimate (2.5) and Proposition 2.8. Suppose that $a \in SD_p^s(w)$ with $p \geq rs$. Then, for every $s_0 \geq s$,

$$\sum_j a(Q_j)^p w(Q_j) \leq \|a\|^p_{SD_p^s(w)} \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q)$$
$$\leq \|a\|^p_{SD_p^s(w)} \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p}{s_0}} a(Q)^p w(Q),$$

for any family $\{Q_j\}_j$ of pairwise disjoint subcubes of the cube Q. Hence $\|a\|_{SD_p^{s_0}(w)} \leq \|a\|_{SD_p^s(w)}$. For each $p < q$, we choose s_0 to be the constant satisfying the relation

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{rs_0}.$$

Since $p < rs_0$, applying estimate (2.5) gives

$$\left\|f - f_Q\right\|_{L^{q,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq c_n q [w]_{A_\infty} [w]^{\frac{1}{p}}_{A_r} \|a\|_{SD_p^s(w)} a(Q),$$

for every $q > p \geq 1$. We can apply Proposition 2.8 with

$$\gamma = c_n [w]_{A_\infty} [w]^{\frac{1}{p}}_{A_r} \|a\|_{SD_p^s(w)} a(Q),$$

and we obtain the desired result (2.6).

We now prove the estimate (2.5). Fix a cube Q, our goal is to show that for every $t > 0$,

$$t^{p^*_{r,s}} w(\{x \in Q : |f(x) - f_Q| > t\}) \leq \|a\|^{p^*_{r,s}} (c_n p^*_{r,s} [w]_{A_\infty} [w]^{\frac{1}{rs}}_{A_r} a(Q))^{p^*_{r,s}} w(Q). \tag{2.8}$$

Denote $\mathcal{M}_Q$ as the dyadic maximal operator localized in Q. Since

$$|f(x) - f_Q| \leq \mathcal{M}_Q(f - f_Q)(x),$$

we can estimate the larger set

$$\Omega_t = \{x \in Q : \mathcal{M}_Q(f - f_Q)(x) > t\}.$$

Let $\{Q_j\}_j$ be the maximal dyadic cubes forming Ω_t. Then, either $\{Q_j\}_j = \{Q\}$ or the cubes Q_j satisfy the condition

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f - f_Q| \leq 2^n t \tag{2.9}$$

and for almost every $x \in Q \setminus \cup_j Q_j$, we have $|f(x) - f_Q| \leq t$. Let $\kappa = 2^n + 1$. We claim that

$$w(\Omega_{\kappa t}) \leq \sum_j w(E_{Q_j}),$$

where

$$E_{Q_j} = \{x \in Q_j : \mathcal{M}_{Q_j}(f - f_{Q_j})(x) > t\}.$$

When $\{Q_j\}_j = \{Q\}$, this claim holds trivially. In the other case, since $\kappa > 1$, it is not difficult to show that, except for a set of null measure,

$$\Omega_{\kappa t} \subset \bigcup_j \left\{x \in Q_j : \mathcal{M}_{Q_j}(f - f_{Q_j})(x) > t\right\}.$$

For every integer j, by (2.9), we have $|f_{Q_j} - f_Q| \leq 2^n t$. Now let $x \in \Omega_{\kappa t} \cap Q_j$. Since $\kappa > 1$ and Q_j is maximal in the Calderón–Zygmund decomposition of Ω_t, the supremum defining $\mathcal{M}_Q(f - f_Q)(x)$ may be restricted to dyadic subcubes of Q_j. Hence

$$\begin{aligned}
\kappa t &< \mathcal{M}_Q(f - f_Q)(x) \\
&= \mathcal{M}_{Q_j}(f - f_Q)(x) \\
&\leq \mathcal{M}_{Q_j}(f - f_{Q_j})(x) + |f_{Q_j} - f_Q| \\
&\leq \mathcal{M}_{Q_j}(f - f_{Q_j})(x) + 2^n t.
\end{aligned}$$

Since $\kappa = 2^n + 1$, it follows that

$$\mathcal{M}_{Q_j}(f - f_{Q_j})(x) > t,$$

and this completes the proof of the claim. Note that for every $\gamma > 0$,

$$\begin{aligned}
E_{Q_j} \subseteq &\{x \in Q_j : \mathcal{M}_{Q_j}(f - f_{Q_j})(x) > t,\ \mathcal{M}_Q^{\#} f(x) \leq \gamma t\} \cup \{x \in Q_j : \mathcal{M}_Q^{\#} f(x) > \gamma t\} \\
:= &A_j \cup B_j.
\end{aligned}$$

Thus, we have

$$w(\Omega_{\kappa t}) \leq \sum_j w(E_{Q_j}) \leq \sum_j w(A_j) + \sum_j w(B_j).$$

For the sets A_j, we use Proposition 2.9,

$$\sum_j w(A_j) \leq c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} \sum_j w(Q_j) = c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} w(\Omega_t),$$

where $c_1, c_2 > 0$ are dimensional constants. On the other hand, for B_j we can argue as follows. We have

$$\bigcup_j B_j \subseteq \{x \in Q : \mathcal{M}_Q^{\#} f(x) > \gamma t\} := \bigcup_i R_i,$$

where R_i are the maximal dyadic subcubes of Q such that

$$\gamma t < \frac{1}{|R_i|} \int_{R_i} |f(x) - f_{R_i}| dx.$$

By the initial hypothesis of the theorem, we know that $\gamma t < a(R_i)$. Now, we will be using the condition that a satisfies, namely $a \in SD_p^s(w)$. Then,

$$\begin{aligned}\sum_j w(B_j) \leq & w\big(\{x \in Q : \mathcal{M}_Q^{\#} f(x) > \gamma t\}\big) \\ &= \sum_i w(R_i) \\ &\leq \left(\frac{1}{\gamma t}\right)^p \sum_i a(R_i)^p w(R_i) \\ &\leq \|a\|^p \left(\frac{1}{\gamma t}\right)^p \left(\frac{|\bigcup_i R_i|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q) \\ &\leq \|a\|^p [w]_{A_r}^{\frac{p}{rs}} \left(\frac{1}{\gamma t}\right)^p \left(\frac{w(\bigcup_i R_i)}{w(Q)}\right)^{\frac{p}{rs}} a(Q)^p w(Q).\end{aligned} \tag{2.10}$$

This implies that

$$\left(\frac{w(\bigcup_i R_i)}{w(Q)}\right)^{1-\frac{p}{rs}} \leq [w]_{A_r}^{\frac{p}{rs}} \left(\frac{1}{\gamma t}\right)^p \|a\|^p a(Q)^p,$$

and hence

$$\frac{w(\bigcup_i R_i)}{w(Q)} \leq [w]_{A_r}^{\frac{p_{r,s}^*}{rs}} \left(\frac{1}{\gamma t}\right)^{p_{r,s}^*} \|a\|^{p_{r,s}^*} a(Q)^{p_{r,s}^*}.$$

Combining these results, it follows that,

$$w(\Omega_{\kappa t}) \leq c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} w(\Omega_t) + [w]_{A_r}^{\frac{p_{r,s}^*}{rs}} \left(\frac{1}{\gamma t}\right)^{p_{r,s}^*} \|a\|^{p_{r,s}^*} a(Q)^{p_{r,s}^*} w(Q)$$

and then

$$(\kappa t)^{p_{r,s}^*} w(\Omega_{\kappa t}) \leq c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} (\kappa t)^{p_{r,s}^*} w(\Omega_t) + [w]_{A_r}^{\frac{p_{r,s}^*}{rs}} \left(\frac{\kappa}{\gamma}\right)^{p_{r,s}^*} \|a\|^{p_{r,s}^*} a(Q)^{p_{r,s}^*} w(Q).$$

We define

$$\varphi(N) = \sup_{0<t\leq N} t^{p_{r,s}^*} w(\Omega_t),$$

since this function is increasing, we have

$$\varphi(N) \leq \varphi(N\kappa) \leq c_1 \kappa^{p_{r,s}^*} e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} \varphi(N) + [w]_{A_r}^{\frac{p_{r,s}^*}{rs}} \left(\frac{\kappa}{\gamma}\right)^{p_{r,s}^*} \|a\|^{p_{r,s}^*} a(Q)^{p_{r,s}^*} w(Q).$$

Since the parameter γ remains free throughout the derivation, we may set it such that

$$c_1 \kappa^{p_{r,s}^*} e^{-\frac{c_2}{\gamma[w]_{A_\infty}}} = \frac{1}{2},$$

then

$$\frac{1}{\gamma} \leq c_n [w]_{A_\infty} p_{r,s}^*.$$

This yields the estimate

$$\varphi(N) \leq c_n^{p_{r,s}^*} [w]_{A_\infty}^{p_{r,s}^*} [w]_{A_r}^{\frac{p_{r,s}^*}{rs}} (p_{r,s}^*)^{p_{r,s}^*} \|a\|^{p_{r,s}^*} a(Q)^{p_{r,s}^*} w(Q).$$

Passing to the limit $N \to \infty$ in the above inequality yields (2.8), concluding the proof of the theorem. □

Before addressing the sharpness of Theorem 2.5, we present a consequence of Proposition 2.8. Specifically, the following result shows that the condition $w \in A_\infty$ is necessary for the D_p-type self-improving property (Theorem 2.2) to hold. More precisely, if the conclusion of Theorem 2.2 holds for every functional $a \in D_p(w)$, then w must belong to A_∞, with quantitative control of $[w]_{A_\infty}$.

Proposition 2.10. *The assumption $w \in A_\infty$ in Theorem 2.2 is necessary. More precisely, let w be a weight and assume that there exists a constant $C_w < \infty$ such that, for every $1 \le p < \infty$, every functional $a \in D_p(w)$, and every locally integrable function f satisfying*

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le a(Q),$$

for every cube Q, we have

$$\|f - f_Q\|_{L^{p,\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \le C_w p\ \|a\|_{D_p(w)}\, a(Q), \tag{2.11}$$

for every cube Q. Then $w \in A_\infty$ with $[w]_{A_\infty} \le c_n C_w$.

Proof. Assume that (2.11) holds for a weight w. Let $f \in \mathrm{BMO}$ and define $a(Q) := \|f\|_{\mathrm{BMO}}$ for every cube Q. Then,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le a(Q),$$

for every cube Q. Moreover, $a \in D_p(w)$ for every $1 \le p < \infty$ and $\|a\|_{D_p(w)} = 1$. Applying (2.11), we obtain, for every cube Q and every $1 < p < \infty$,

$$\|f - f_Q\|_{L^{p,\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \le C_w p\, \|f\|_{\mathrm{BMO}}.$$

By Proposition 2.8, it follows that

$$\|f - f_Q\|_{\exp L\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \le C\, C_w\, \|f\|_{\mathrm{BMO}}.$$

Hence $f \in \mathrm{BMO}_w$ and

$$\|f\|_{\mathrm{BMO}_w} \le C\, C_w\, \|f\|_{\mathrm{BMO}}. \tag{2.12}$$

Thus we have proved $\mathrm{Id} : \mathrm{BMO} \hookrightarrow \mathrm{BMO}_w$. However, by the characterization proved in **[OPRR20, Theorem 1.2]**, the embedding $\mathrm{Id} : \mathrm{BMO} \hookrightarrow \mathrm{BMO}_w$ holds if and only if $w \in A_\infty$, therefore $w \in A_\infty$. Moreover, taking the supremum in (2.12) over all $f \in \mathrm{BMO}$ with $\|f\|_{\mathrm{BMO}} = 1$, and using again **[OPRR20, Theorem 1.2]**, we obtain $[w]_{A_\infty} \le c_n C_w$. □

2.2.1 Sharpness of the exponent

We next show that the exponent $p^*_{r,s}$ in (2.5) is optimal, in the sense that it cannot in general be replaced by any larger exponent. The examples below cover the range $s \ge n$, $1 \le r \le p$, and $p < rs$.

We start with the case $s = n$ and $1 \le r \le p < nr$. Since $A_r \subset A_p$, the weighted Poincaré inequality obtained from (2.1) and (1.8) applies to every $w \in A_r$. Thus

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le c_n [w]_{A_p}^{\frac{1}{p}}\, \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)\,dx\right)^{\frac{1}{p}} := a(Q). \tag{2.13}$$

Moreover, from **[PR19, Lemma 3.2]**, we know that $a \in SD_p^n(w)$. Then, applying Theorem 2.5, we obtain

$$\left\|f - f_Q\right\|_{L^{p_r^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \leq c_n p_r^* [w]_{A_\infty}\, [w]_{A_p}^{\frac{1}{p}}\, [w]_{A_r}^{\frac{1}{rn}}\, \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p\, w(x)\,dx\right)^{\frac{1}{p}},$$

for every cube Q, where $\frac{1}{p_r^*} = \frac{1}{p} - \frac{1}{nr}$. The next proposition shows that this exponent cannot be improved.

For the sake of simplicity, in the following two propositions, we use the notation $A \lesssim B$ to indicate that there exists a constant C, depending on n, p, q and $[w]_{A_r}$, and on any fixed auxiliary cut-off function chosen in the proof, such that $A \leq CB$.

Proposition 2.11. *Suppose $1 \leq r \leq p$ and $0 < \frac{1}{q} < \frac{1}{p} - \frac{1}{nr}$. Let $Q = [-1,1]^n$. Then there exists a weight $w \in A_r$ such that*

$$\left\|f - f_Q\right\|_{L^{q,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \lesssim \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p\, w(x)\,dx\right)^{\frac{1}{p}} \tag{2.14}$$

cannot hold for every $f \in C_c^\infty(\mathbb{R}^n)$.

Proof. Suppose (2.14) holds for every $f \in C_c^\infty(\mathbb{R}^n)$. Let $B = B(0,1)$. Since A_p weights satisfy the doubling condition and $B \subset Q \subset B(0,\sqrt{n})$, applying (2.1) and (2.13), for every $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
\|f - f_B\|_{L^{q,\infty}\left(B,\frac{w\,dx}{w(B)}\right)} &\leq \|f - f_Q\|_{L^{q,\infty}\left(B,\frac{w\,dx}{w(B)}\right)} + |f_B - f_Q| \\
&\lesssim \|f - f_Q\|_{L^{q,\infty}\left(Q,\frac{w\,dx}{w(Q)}\right)} + \frac{1}{|Q|}\int_Q |f - f_Q|\,dx \\
&\lesssim \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\,dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{2.15}$$

Let $w(x) = |x|^{\delta - n}$ with $0 < \delta < nr$, it is well-known that w is an A_r weight. Let φ be a smooth function supported in $B(0,2)$ and equal to 1 on $B(0,1)$. For $0 < \varepsilon < \frac{1}{4}$, we define $f_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$. Since f_ε vanishes on $E = B(0,1)\backslash B(0,\frac{1}{2})$, applying (2.15),

$$\begin{aligned}
\|f_\varepsilon\|_{L^{q,\infty}\left(B,\frac{w\,dx}{w(B)}\right)} &\leq \|f_\varepsilon - (f_\varepsilon)_B\|_{L^{q,\infty}\left(B,\frac{w\,dx}{w(B)}\right)} + |(f_\varepsilon)_B| \\
&\leq \|f_\varepsilon - (f_\varepsilon)_B\|_{L^{q,\infty}\left(B,\frac{w\,dx}{w(B)}\right)} + \|f_\varepsilon - (f_\varepsilon)_B\|_{L^{q,\infty}\left(E,\frac{w\,dx}{w(E)}\right)} \\
&\lesssim \|f_\varepsilon - (f_\varepsilon)_B\|_{L^{q,\infty}\left(B,\frac{w\,dx}{w(B)}\right)} \\
&\lesssim \left(\frac{1}{w(Q)}\int_Q |\nabla f_\varepsilon|^p w\,dx\right)^{\frac{1}{p}},
\end{aligned} \tag{2.16}$$

where we used that $f_\varepsilon = 0$ on E, and hence

$$|(f_\varepsilon)_B| \leq \|f_\varepsilon - (f_\varepsilon)_B\|_{L^{q,\infty}\left(E,\frac{w\,dx}{w(E)}\right)},$$

together with the fact that $w(B(0,1)) \simeq \frac{1}{\delta} \simeq w(E)$. Note that $f_\varepsilon = 1$ on $B(0,\varepsilon)$ and $|\nabla f_\varepsilon| \lesssim \frac{1}{\varepsilon}\chi_{B(0,2\varepsilon)}$. Therefore,

$$\|f_\varepsilon\|_{L^{q,\infty}\left(B,\frac{w\,dx}{w(B)}\right)} \geq \left(\frac{w(B(0,\varepsilon))}{w(B)}\right)^{\frac{1}{q}} \simeq \varepsilon^{\delta/q},$$

and

$$\left(\frac{1}{w(Q)}\int_Q |\nabla f_\varepsilon|^p w\,dx\right)^{\frac{1}{p}} \lesssim \frac{1}{\varepsilon}\left(\frac{w(B(0,2\varepsilon))}{w(Q)}\right)^{\frac{1}{p}} \simeq \varepsilon^{\delta/p-1}. \tag{2.17}$$

Applying estimate (2.16), we have

$$\varepsilon^{\delta(\frac{1}{q}-\frac{1}{p})+1} \lesssim 1. \tag{2.18}$$

Since $\frac{1}{q}-\frac{1}{p} < -\frac{1}{nr}$, we can choose δ sufficiently close to nr so that

$$\delta\left(\frac{1}{q}-\frac{1}{p}\right) < -1.$$

Then the exponent in (2.18) is negative, so the left-hand side tends to infinity as $\varepsilon \to 0$, contradicting (2.18). □

Now, we construct the corresponding counterexample for the case $s > n$ and $1 \le r \le p < rs$. As before, since $A_r \subset A_p$, applying the fractional Poincaré inequality (1.19) and (1.8), for $0 < \alpha < 1$ and $w \in A_r$,

$$\frac{1}{|Q|}\int_Q |f - f_Q|\,dx \lesssim \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |\mathcal{D}_Q^\alpha f(x)|^p w(x)\,dx\right)^{\frac{1}{p}} := a(Q),$$

where

$$\mathcal{D}_Q^\alpha f(x) := \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dy$$

is the local fractional derivative of order α. From **[PR19, Lemma 3.2]**, we know that $a \in SD_p^s(w)$ with $s = \frac{n}{\alpha} > n$. Then, applying Theorem 2.5, we obtain

$$\left\|f - f_Q\right\|_{L^{p_r^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \lesssim \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |\mathcal{D}_Q^\alpha f(x)|^p w(x)\,dx\right)^{\frac{1}{p}},$$

where $\frac{1}{p_r^*} = \frac{1}{p} - \frac{1}{sr}$. The following proposition proves the sharpness of (2.5) in this fractional regime.

Proposition 2.12. *Suppose $1 \le r \le p$ and $0 < \frac{1}{q} < \frac{1}{p} - \frac{\alpha}{nr}$. Let $Q = [-1,1]^n$. Then there exists a weight $w \in A_r$ such that*

$$\left\|f - f_Q\right\|_{L^{q,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \lesssim \left(\frac{1}{w(Q)}\int_Q |\mathcal{D}_Q^\alpha f(x)|^p w(x)\,dx\right)^{\frac{1}{p}} \tag{2.19}$$

cannot hold for every $f \in C_c^\infty(\mathbb{R}^n)$.

Proof. The strategy of the proof is similar to that of Proposition 2.11; the only difference appears in the estimate (2.17). We use the same weight

$$w(x) = |x|^{\delta-n},$$

with $0 < \delta < nr$, and the same test functions $f_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$. We decompose

$$\int_{2B} |\mathcal{D}_{2B}^\alpha f_\varepsilon|^p |x|^{\delta-n}\,dx \le \int_{B(0,4\varepsilon)} |\mathcal{D}_{2B}^\alpha f_\varepsilon|^p |x|^{\delta-n}\,dx + \sum_{k=2}^{\lfloor \log_2(1/\varepsilon)\rfloor} \int_{B(0,2^{k+1}\varepsilon)\setminus B(0,2^k\varepsilon)} |\mathcal{D}_{2B}^\alpha f_\varepsilon|^p |x|^{\delta-n}\,dx.$$

For $x \in B(0,4\varepsilon)$,

$$\begin{aligned}
\mathcal{D}^{\alpha}_{2B} f_\varepsilon(x) &= \int_{2B} \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x-y|^{n+\alpha}}\, dy \\
&= \int_{B(x,4\varepsilon)} \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x-y|^{n+\alpha}}\, dy + \int_{2B\setminus B(x,4\varepsilon)} \frac{|f_\varepsilon(x) - f_\varepsilon(y)|}{|x-y|^{n+\alpha}}\, dy \\
&\le \frac{1}{\varepsilon}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(x,4\varepsilon)} \frac{|x-y|}{|x-y|^{n+\alpha}}\, dy + \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n\setminus B(x,4\varepsilon)} \frac{1}{|x-y|^{n+\alpha}}\, dy \\
&\le \frac{1}{\varepsilon}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,4\varepsilon)} |y|^{-n-\alpha+1}\, dy + \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n\setminus B(0,4\varepsilon)} |y|^{-n-\alpha}\, dy \\
&\lesssim \varepsilon^{-\alpha}.
\end{aligned}$$

For $x \in B(0,2^{k+1}\varepsilon)\setminus B(0,2^k\varepsilon)$ with $k \ge 2$, we have $f_\varepsilon(x) = 0$ and $|x-y| \simeq |x| \simeq 2^k\varepsilon$ whenever $y \in \operatorname{supp} f_\varepsilon$. Thus,

$$\mathcal{D}^{\alpha}_{2B} f_\varepsilon(x) = \int_{B(0,2\varepsilon)} \frac{|f_\varepsilon(y)|}{|x-y|^{n+\alpha}}\, dy \lesssim \|\varphi\|_{L^\infty(\mathbb{R}^n)} (2^k\varepsilon)^{-n-\alpha} (2\varepsilon)^n \lesssim (2^k)^{-n-\alpha}\varepsilon^{-\alpha}.$$

Combining everything together,

$$\begin{aligned}
&\left(\frac{1}{w(2B)} \int_{2B} |\mathcal{D}^{\alpha}_{2B} f_\varepsilon|^p |x|^{\delta-n}\, dx\right)^{\frac{1}{p}} \\
&\lesssim \delta^{\frac{1}{p}} \left(\varepsilon^{-\alpha p} \int_{B(0,4\varepsilon)} |x|^{\delta-n}\, dx + \varepsilon^{-\alpha p} \sum_{k=2}^{\lfloor \log_2(1/\varepsilon)\rfloor} (2^k)^{-np-\alpha p} \int_{B(0,2^{k+1}\varepsilon)\setminus B(0,2^k\varepsilon)} |x|^{\delta-n}\, dx\right)^{\frac{1}{p}} \\
&\lesssim \varepsilon^{-\alpha} \left(\varepsilon^{\delta} + \varepsilon^{\delta} \sum_{k=2}^{\lfloor \log_2(1/\varepsilon)\rfloor} (2^k)^{-np-\alpha p+\delta}\right)^{\frac{1}{p}} \\
&\lesssim \varepsilon^{\delta/p-\alpha}.
\end{aligned} \tag{2.20}$$

On the other hand, as in the proof of Proposition 2.11,

$$\|f_\varepsilon\|_{L^{q,\infty}\left(B, \frac{w\,dx}{w(B)}\right)} \gtrsim \left(\frac{w(B(0,\varepsilon))}{w(B)}\right)^{\frac{1}{q}} \simeq \varepsilon^{\delta/q}.$$

Hence, if (2.19) were true, then (2.20) would imply

$$\varepsilon^{\delta/q} \lesssim \varepsilon^{\delta/p-\alpha},$$

or equivalently

$$\varepsilon^{\delta(\frac{1}{q}-\frac{1}{p})+\alpha} \lesssim 1.$$

Since $\frac{1}{q} - \frac{1}{p} < -\frac{\alpha}{nr}$, we can choose δ sufficiently close to nr such that

$$\delta\left(\frac{1}{q} - \frac{1}{p}\right) < -\alpha.$$

Then the last estimate fails as $\varepsilon \to 0$, and we arrive at a contradiction. □

2.3 Self-improving results to $L^{p(1+\frac{1}{s}),\infty}$ generalized Poincaré-type inequalities

For the A_∞ class of weights, we refine Theorems 2.2 and 2.4 by obtaining a stronger self-improving conclusion. More precisely, we show that, under the condition $SD_p^s(w)$, the corresponding Poincaré-type estimate improves to an estimate in $L^{p(1+\frac{1}{s}),\infty}$ with respect to the normalized weighted measure.

Theorem 2.13. *Let $w \in A_\infty$, let $1 < p < \infty$ and let a be a functional over cubes. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally integrable function such that*

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \tag{2.21}$$

for every cube Q. If $a \in SD_p^s(w)$ for $s > 0$, then for every cube Q,

$$\left\|f - f_Q\right\|_{L^{p\left(1+\frac{1}{s}\right),\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq c_n p\left(1+\frac{1}{s}\right)[w]_{A_\infty}\|a\|^{\frac{s}{s+1}}\, a(Q).$$

Remark 2.14. In the limiting case $s = \infty$, the condition $SD_p^s(w)$ formally reduces to $D_p(w)$. Thus Theorem 2.13 recovers Theorem 2.2, with the same dependence on the constants.

Comparing with the estimate (2.5), Theorem 2.13 provides a better result under the assumption $r > p\left(1+\frac{1}{s}\right)$, in the sense that the norm on the left-hand side is larger.

Proof of Theorem 2.13. The proof strategy is similar to that of Theorem 2.5; we will only highlight the differences. Starting from (2.10), using the triangle inequality and the definition of the averages, we obtain

$$\gamma t < \frac{1}{|R_i|}\int_{R_i}|f - f_{R_i}| \leq \frac{1}{|R_i|}\int_{R_i}|f - f_Q| + |f_Q - f_{R_i}| \leq 2\frac{1}{|R_i|}\int_{R_i}|f - f_Q|.$$

Hence,

$$\left|\bigcup_i R_i\right| = \sum_i |R_i| < \frac{2}{\gamma t}\sum_i \int_{R_i}|f - f_Q| \leq \frac{2}{\gamma t}\int_Q |f - f_Q| \leq \frac{2}{\gamma t}a(Q)|Q|.$$

Then,

$$\begin{aligned}\sum_i w(R_i) \leq& \|a\|^p\left(\frac{1}{\gamma t}\right)^p\left(\frac{2}{\gamma t}a(Q)\right)^{\frac{p}{s}} w(Q)a(Q)^p\\ =&\|a\|^p 2^{\frac{p}{s}}\left(\frac{1}{\gamma t}a(Q)\right)^{p+\frac{p}{s}} w(Q).\end{aligned}$$

We proceed as before, combining everything together. Let $\kappa = 2^n + 1$. Then

$$w(\Omega_{\kappa t}) \leq c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} w(\Omega_t) + \|a\|^p 2^{\frac{p}{s}}\left(\frac{1}{\gamma t}a(Q)\right)^{p+\frac{p}{s}} w(Q),$$

and then

$$(\kappa t)^{p+\frac{p}{s}} w(\Omega_{\kappa t}) \leq c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}}(\kappa t)^{p+\frac{p}{s}} w(\Omega_t) + \|a\|^p\left(\frac{\kappa}{\gamma}\right)^{p+\frac{p}{s}} 2^{\frac{p}{s}} w(Q)a(Q)^{p+\frac{p}{s}}.$$

Defining

$$\varphi(N) = \sup_{0<t\leq N} t^{p+\frac{p}{s}} w(\Omega_t)$$

and choosing γ satisfying

$$c_1 \kappa^{p+\frac{p}{s}} e^{-\frac{c_2}{\gamma [w]_{A_\infty}}} = \frac{1}{2},$$

we obtain that

$$\varphi(N) \leq c_n^{p+\frac{p}{s}} [w]_{A_\infty}^{p+\frac{p}{s}} \left(p(1+\frac{1}{s})\right)^{p+\frac{p}{s}} \|a\|^p w(Q) a(Q)^{p+\frac{p}{s}}.$$

Letting N tend to ∞, we obtain the desired result. □

We also provide a self-improving result for *general* weights.

Theorem 2.15. *Let $0 < p < \infty$ and let w be any weight. Consider a functional a satisfying the $SD_p^s(w)$ condition with $s > 0$ and constant $\|a\|$. Let f be a locally integrable function such that*

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq a(Q) \tag{2.22}$$

and

$$\left\| f - f_Q \right\|_{L^{p,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq K a(Q), \tag{2.23}$$

for every cube Q, where K is a constant independent of Q. Then, there exists a dimensional constant $c_n > 0$ such that for any cube Q,

$$\left\| f - f_Q \right\|_{L^{p\left(1+\frac{1}{s}\right),\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq c_n \max\left\{1, (K\|a\|)^{\frac{s}{s+1}}\right\} a(Q).$$

Condition (2.23) is fairly mild and holds for a wide range of practical cases. Indeed, there are at least two standard ways to verify it. First, if $w \in A_\infty$, one may invoke Theorem 2.2 to obtain a weighted weak-L^p estimate. This yields a conclusion of the same qualitative nature as Theorem 2.13, albeit with a better dependence on $[w]_{A_\infty}$ (and, in contrast, a worse dependence on $\|a\|$). Second, for completely general weights, one may appeal to Theorem 2.4, which provides a weighted strong L^p generalized Poincaré inequality. This immediately implies (2.23) with $K = c_n(1+s)\|a\|$. Therefore, Theorem 2.15 applies under very mild additional input, and as an immediate corollary, we recover and sharpen the main result of **[PR19]** and its recent refinement in **[LLO22]**.

Now, we turn to the proof of Theorem 2.15, which covers the case of general weights.

Proof of Theorem 2.15. Let $0 < p < \infty$. Fix a cube Q and let $t > a(Q)$. Observe that by (2.22) we have,

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq a(Q) < t.$$

We consider the local Calderón–Zygmund decomposition of the function $|f - f_Q|$ relative to the cube Q at height t. Then, there is a family of dyadic pairwise disjoint subcubes $\{Q_j\}_j$ with respect to Q, which satisfy the following properties. For each j,

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f(x) - f_Q| dx \leq 2^n t, \tag{2.24}$$

and for almost every $x \in Q \setminus \bigcup_j Q_j$,

$$|f(x) - f_Q| \le t.$$

From the first initial hypothesis (2.22) and (2.24),

$$\frac{\left|\bigcup_j Q_j\right|}{|Q|} \le \frac{1}{t|Q|}\int_Q |f - f_Q|\,dx \le \frac{a(Q)}{t}. \tag{2.25}$$

Let $\kappa = 2^n + 1$. Since (2.24) implies that $|f_{Q_j} - f_Q| \le 2^n t$, we have,

$$\begin{aligned} w\left(\left\{x \in Q : |f(x) - f_Q| > \kappa t\right\}\right) &\le \sum_j w\left(\left\{x \in Q_j : |f(x) - f_{Q_j}| + |f_{Q_j} - f_Q| > \kappa t\right\}\right) \\ &\le \sum_j w\left(\left\{x \in Q_j : |f(x) - f_{Q_j}| > t\right\}\right). \end{aligned}$$

We can estimate the sum using the second hypothesis (2.23) at each cube Q_j and then the fact that the functional a satisfies $SD_p^s(w)$ condition,

$$\begin{aligned} w\left(\left\{x \in Q : |f(x) - f_Q| > \kappa t\right\}\right) &\le \sum_j w\left(\left\{x \in Q_j : |f(x) - f_{Q_j}| > t\right\}\right) \\ &\le K^p t^{-p} \sum_j a(Q_j)^p w(Q_j) \\ &\le K^p t^{-p} \|a\|^p \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q) \\ &\le K^p \|a\|^p \left(\frac{a(Q)}{t}\right)^{p+\frac{p}{s}} w(Q), \end{aligned}$$

where in the last inequality, we used (2.25). Combining these estimates, we have proved

$$t\, w\left(\left\{x \in Q : |f(x) - f_Q| > \kappa t\right\}\right)^{\frac{1}{p+\frac{p}{s}}} \le (K\|a\|)^{\frac{s}{s+1}} w(Q)^{\frac{1}{p+\frac{p}{s}}} a(Q),$$

for all $t > a(Q)$. This is equivalent to saying that

$$\sup_{t > c_n a(Q)} t\left(\frac{1}{w(Q)} w\left(\left\{x \in Q : |f(x) - f_Q| > t\right\}\right)\right)^{\frac{1}{p+\frac{p}{s}}} \le c_n (K\|a\|)^{\frac{s}{s+1}} a(Q). \tag{2.26}$$

On the other hand,

$$\sup_{t \le c_n a(Q)} t\left(\frac{1}{w(Q)} w\left(\left\{x \in Q : |f(x) - f_Q| > t\right\}\right)\right)^{\frac{1}{p+\frac{p}{s}}} \le c_n a(Q).$$

Hence,

$$\begin{aligned} \left\|f - f_Q\right\|_{L^{p+\frac{p}{s},\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} &= \sup_{t>0} t\left(\frac{1}{w(Q)} w\left(\left\{x \in Q : |f(x) - f_Q| > t\right\}\right)\right)^{\frac{1}{p+\frac{p}{s}}} \\ &= \max\left\{\sup_{t \le c_n a(Q)} \cdots, \sup_{t > c_n a(Q)} \cdots\right\} \\ &\le \max\left\{c_n a(Q), c_n (K\|a\|)^{\frac{s}{s+1}} a(Q)\right\} \\ &= c_n \max\left\{1, (K\|a\|)^{\frac{s}{s+1}}\right\} a(Q). \end{aligned}$$

This concludes the proof. □

As an immediate consequence of the argument, we also obtain a self-improvement along the Lorentz scale, using (1.4) and the previous result we have the following corollary.

Corollary 2.16. *Let $0 < p < \infty$ and let w be any weight. Consider a functional a satisfying the $SD_p^s(w)$ condition with $s > 0$ and constant $\|a\|$. Let f be a locally integrable function such that*

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le a(Q)$$

and

$$\left\|f - f_Q\right\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le Ka(Q),$$

for every cube Q, where K is a constant independent of Q. Then there exists a dimensional constant $c_n > 0$ such that, for every cube Q and every $1 \le q < \infty$,

$$\left\|f - f_Q\right\|_{L^{p,q}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le c_n \left(\frac{p}{q}\right)^{\frac{1}{q}} \max\left\{1, (K\|a\|)^{\frac{s}{s+1}}\right\} (1+s)^{\frac{1}{q}}\, a(Q). \tag{2.27}$$

Remark 2.17. In particular, within this framework, the weak estimate (2.23) is, up to quantitative constants, equivalent to the family of strong Lorentz estimates (2.27). This is noteworthy since the endpoint $q = 1$ is a priori much stronger than the weak endpoint $q = \infty$.

2.4 Applications to weighted Poincaré–Sobolev inequalities

In this section we apply the self-improving results proved in the previous sections to prove weighted Poincaré–Sobolev inequalities.

2.4.1 Weak and strong Poincaré–Sobolev inequalities

As a first application of Theorem 2.5, we recover the following weighted weak Poincaré–Sobolev estimate.

Theorem 2.18. *Let $w \in A_r$ with $1 \le r \le p$ and $1 \le p < nr$. Let p_r^* be defined by*

$$\frac{1}{p} - \frac{1}{p_r^*} = \frac{1}{n}\frac{1}{r}. \tag{2.28}$$

Then there exists a dimensional constant $c_n > 0$ such that, for every cube Q,

$$\left\|f - f_Q\right\|_{L^{p_r^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le c_n p_r^* [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{rn}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p\, w(x)\,dx\right)^{\frac{1}{p}}. \tag{2.29}$$

Proof. Since $1 \le r \le p$, we have $A_r \subset A_p$. By using (2.1) and (1.8), we obtain

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le c_n [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)\,dx\right)^{\frac{1}{p}} := a(Q).$$

Moreover, from **[PR19, Lemma 3.2]**, we know that $a \in SD_p^n(w)$. Applying Theorem 2.5 with $s = n$, we obtain (2.29). □

Remark 2.19. The exponent p_r^* in (2.29) is sharp within the class A_r. We have proved this in Proposition 2.11 (see also Proposition 3.25 for a related sharpness result).

The weak estimate (2.29) can be upgraded to a strong Lorentz estimate by means of the truncation argument.

Theorem 2.20. *Let $w \in A_r$ with $1 \le r \le p$, $1 \le p < nr$ and consider p_r^* defined in* (2.28)*. Then, there exists a dimensional constant $c_n > 0$ such that, for every cube Q,*

$$\|f - f_Q\|_{L^{p_r^*,p}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le c_n p_r^* [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{rn}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p \, w(x)\, dx\right)^{\frac{1}{p}}. \quad (2.30)$$

Proof. The proof follows by applying the truncation argument stated in Theorem 1.35. □

Remark 2.21. The second Lorentz index p in the target space $L^{p_r^*,p}$ in (2.30) is natural from the viewpoint of Sobolev-Lorentz embeddings. Already in the unweighted case $w = 1$ (hence $r = 1$ and $p_r^* = p^*$), the endpoint embedding $W^{1,p} \hookrightarrow L^{p^*,p}$ is optimal along the Lorentz scale (and, more generally, within rearrangement-invariant spaces), in the sense that $W^{1,p} \not\hookrightarrow L^{p^*,q}$ for any $q < p$ (see **[EKP00]**). Consequently, we should not expect a general improvement of (2.30) with $L^{p_r^*,q}$, $q < p$, even in the simplest situation.

Remark 2.22. We remark that we could bound $[w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{rn}}$ by a multiple of $[w]_{A_r}^{1+\frac{1}{p}+\frac{1}{rn}}$.

As estimates (2.29) and (2.30) are derived from the self-improving argument under a very general setting, they are not sharp in the A_r constant. The optimal A_r dependence in this scale was recently identified in **[Cla25a]**, where it is shown that the best possible exponent is $[w]_{A_r}^{1/p}$. We will show this sharp dependence in Chapter 3 by means of different techniques.

We have the following analogue of Theorem 2.20 for pairs of weights (see Definition 1.23).

Proposition 2.23. *Let $(u, v) \in A_p$ and $u \in A_r$ with $1 \le r \le p$. Let p_r^* be defined in* (2.28)*. Assume that $1 \le p < nr$. Then there exists a dimensional constant $c_n > 0$ such that,*

$$\|f - f_Q\|_{L^{p_r^*,p}\left(Q,\frac{u(x)\,dx}{u(Q)}\right)} \le c_n p_r^* [u]_{A_\infty} [u]_{A_r}^{\frac{1}{rn}} [u, v]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{u(Q)} \int_Q |\nabla f(x)|^p \, v(x)\, dx\right)^{\frac{1}{p}},$$

for every cube Q.

2.4.2 Lorentz norms of the gradient

We now show that the same self-improving mechanism also applies when the strong L^p norm of the gradient is replaced by a Lorentz norm. The key point is to verify that the corresponding functional satisfies the appropriate D_p or SD_p^s condition.

Lemma 2.24. *Let $w \in A_r$ with $1 \le r \le p$. For $1 \le M \le \infty$, we define p_M^* by the relation*

$$\frac{1}{p} - \frac{1}{p_M^*} = \frac{1}{nrM}.$$

We consider the following functional over cubes:

$$a(Q) = \ell(Q)\,\|g\|_{L^{p,q}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)},$$

where $q \le p_M^*$. Then,

1. *If $M > 1$, then $a \in SD^{nM'}_{p_M^*}(w)$ with $\|a\| \le [w]_{A_r}^{\frac{1}{nrM}}$.*
2. *If $M = 1$, then $p_M^* = p_r^*$ and $a \in D_{p_r^*}(w)$ with $\|a\| \le [w]_{A_r}^{\frac{1}{nr}}$.*

Proof. Let Q be a cube and $\{Q_j\}_j$ a pairwise disjoint family of subcubes of Q. We will use the characterization of the A_r condition given in (1.9),

$$\begin{aligned}
\sum_j a(Q_j)^{p_M^*} w(Q_j) &= \sum_j \ell(Q_j)^{p_M^*}\,\|g\|^{p_M^*}_{L^{p,q}\left(Q_j,\frac{w(x)\,dx}{w(Q_j)}\right)} w(Q_j) \\
&= \sum_j |Q_j|^{\frac{p_M^*}{n}}\,\|g\|^{p_M^*}_{L^{p,q}(Q_j,w)}\, w(Q_j)^{1-\frac{p_M^*}{p}} \\
&= \sum_j |Q_j|^{\frac{p_M^*}{n}} \left(\frac{1}{w(Q_j)}\right)^{\frac{p_M^*}{nrM}} \|g\|^{p_M^*}_{L^{p,q}(Q_j,w)} \\
&= \sum_j |Q_j|^{\frac{p_M^*}{nM'}} \left(\frac{|Q_j|^r}{w(Q_j)}\right)^{\frac{p_M^*}{nrM}} \|g\|^{p_M^*}_{L^{p,q}(Q_j,w)} \\
&\le \left([w]_{A_r}\frac{|Q|^r}{w(Q)}\right)^{\frac{p_M^*}{nrM}} \sum_j |Q_j|^{\frac{p_M^*}{nM'}}\,\|g\|^{p_M^*}_{L^{p,q}(Q_j,w)} \\
&\le \left([w]_{A_r}\frac{|Q|^r}{w(Q)}\right)^{\frac{p_M^*}{nrM}} \left|\bigcup_j Q_j\right|^{\frac{p_M^*}{nM'}} \sum_j \|g\|^{p_M^*}_{L^{p,q}(Q_j,w)}.
\end{aligned}$$

By hypothesis we have $q \le p_M^*$, and therefore we can use Lemma 1.3 with $\alpha = p_M^*$ to obtain

$$\begin{aligned}
\sum_j a(Q_j)^{p_M^*} w(Q_j) &\le \left([w]_{A_r}\frac{|Q|^r}{w(Q)}\right)^{\frac{p_M^*}{nrM}} \left|\bigcup_j Q_j\right|^{\frac{p_M^*}{nM'}} \|g\|^{p_M^*}_{L^{p,q}(Q,w)} \\
&= [w]_{A_r}^{\frac{p_M^*}{nrM}} \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p_M^*}{nM'}} \ell(Q)^{p_M^*}\,\|g\|^{p_M^*}_{L^{p,q}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} w(Q) \\
&= [w]_{A_r}^{\frac{p_M^*}{nrM}} \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p_M^*}{nM'}} a(Q)^{p_M^*} w(Q).
\end{aligned}$$

This proves that $a \in SD^{nM'}_{p_M^*}(w)$ with

$$\|a\| \le [w]_{A_r}^{\frac{1}{nrM}}$$

when $M > 1$. If $M = 1$, then $p_M^* = p_r^*$ and $M' = \infty$. The same argument gives

$$\sum_j a(Q_j)^{p_r^*} w(Q_j) \le [w]_{A_r}^{\frac{p_r^*}{nr}} a(Q)^{p_r^*} w(Q),$$

and hence $a \in D_{p_r^*}(w)$ with

$$\|a\| \le [w]_{A_r}^{\frac{1}{nr}}.$$

□

Remark 2.25. The weighted Poincaré–Sobolev estimate (2.29) provides a convenient example to compare the self-improvement phenomena associated with the conditions $a \in D_{p_r^*}(w)$ and $a \in SD_p^n(w)$. Denote by $a(Q)$ the right-hand side functional in (2.29), and for $M \ge 1$ consider p_M^* defined in Lemma 2.24. We have proved that $a \in D_{p_r^*}(w)$ with $\|a\| \le [w]_{A_r}^{1/(nr)}$ and $a \in SD_{p_M^*}^{nM'}(w)$ with $\|a\| \le [w]_{A_r}^{1/(nrM)}$ for every $M > 1$. Consequently, whether one applies the self-improving result under the $D_{p_r^*}(w)$ condition (Theorem 2.2) or the one under the $SD_{p_M^*}^{nM'}(w)$ condition (Theorem 2.5), one recovers the same conclusion, namely (2.29). This shows the compatibility between the $D_p(w)$ and $SD_p^s(w)$ conditions in this natural model example.

We can now prove the Lorentz refinement with a smaller norm of the gradient.

Theorem 2.26. *Let $w \in A_r$ with $1 \le r \le p$, $1 < p < nr$, and consider p_r^* defined in* (2.28)*. Then, there exists a dimensional constant $c_n > 0$ such that, for every cube Q,*

$$\|f - f_Q\|_{L^{p_r^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le c_n p_r^* [w]_{A_\infty}\, [w]_{A_{p,p_r^*}}^{\frac{1}{p}}\, [w]_{A_r}^{\frac{1}{rn}}\, \ell(Q) \|\nabla f\|_{L^{p,p_r^*}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}. \tag{2.31}$$

Proof. Since $1 \le r \le p$ and $p_r^* > 1$, we have $A_r \subset A_p = A_{p,p_r^*}$. Combining the $(1,1)$ Poincaré inequality (2.1) with the characterization of the $A_{p,q}$ condition given in (1.15), applied with $g = |\nabla f|$, we obtain

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le c_n [w]_{A_{p,p_r^*}}^{\frac{1}{p}}\, \ell(Q)\, \|\nabla f\|_{L^{p,p_r^*}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)},$$

for each cube Q. Now we consider the functional

$$a(Q) = c_n [w]_{A_{p,p_r^*}}^{\frac{1}{p}}\, \ell(Q)\, \|\nabla f\|_{L^{p,p_r^*}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}.$$

By Lemma 2.24 with $M = 1$, we have $a \in D_{p_r^*}(w)$ and $\|a\| \le [w]_{A_r}^{1/(nr)}$. Therefore, applying Theorem 2.2, we obtain (2.31). □

Let us emphasize that the previous argument is specific to the Lorentz norm $\|\cdot\|_{L^{p,p_r^*}}$ on the right-hand side. Indeed, in this setting we have $p_r^* = p_M^*$ when $M = 1$, and this endpoint case corresponds to the condition $D_{p_M^*}(w)$ rather than to an $SD_p^s(w)$-type condition. For this reason, in the proof of Theorem 2.26, we have used Theorem 2.2 instead of Theorem 2.5. By contrast, as we discussed before the proof of Theorem 2.26, if we consider the self-improving result with the norm $\|\cdot\|_{L^p}$ on the right-hand side, then Theorem 2.5 and Theorem 2.2 will lead to the same target exponent, since $(p_M^*)_{r,nM'}^* = p_r^*$.

Remark 2.27. The truncation argument used to strengthen weak-type estimates into Lorentz space bounds (as in Theorem 2.20) is intrinsically related to a summability condition on the right-hand side: it is applicable provided that the Lorentz index satisfies $q \le p$ (see Theorem 1.35). This is consistent with the L^p norm of the gradient term in (2.29), but it does not apply to the smaller Lorentz norm

$\|\nabla f\|_{L^{p,p_r^*}}$ in (2.31), since $p_r^* > p$. Consequently, the same argument cannot strengthen (2.31) to an $L^{p_r^*,p}$ estimate on the left-hand side.

Remark 2.28. We observe that if $w \in A_r$ with $1 < r \le p$, then by combining the previous arguments with the precise open property of the class A_r (see **[HPR12; PR18]**) one can refine the estimates above, in particular improving the target Sobolev exponent beyond p_r^*, namely to $p_{r-\varepsilon}^*$ for some $\varepsilon = \varepsilon(n, r, w) > 0$, as in **[Cla25a]**. We do not pursue this direction here.

2.4.3 The exponential endpoint

In the critical and supercritical range $p \ge rn$, the Sobolev exponent p_r^* is no longer finite, so the L^q estimate must be replaced by an endpoint estimate of exponential type.

Theorem 2.29. *Let $w \in A_r$, let $1 < p < \infty$, and assume that $p \ge nr$. Then, there exists a constant $c_{n,p} > 0$ such that*

$$\|f - f_Q\|_{\exp L\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \le c_{n,p}[w]_{A_\infty}\,[w]_{A_{p,q}}^{\frac{1}{p}}\,[w]_{A_r}^{\frac{1}{p}}\,\ell(Q)\,\|\nabla f\|_{L^{p,q}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)}, \tag{2.32}$$

for every cube Q and every $1 \le q < \infty$.

Proof. Fix $1 \le q < \infty$. Choose $M > 1$ so that $p < nrM$ and $q \le p_M^*$, where p_M^* is defined by

$$\frac{1}{p} - \frac{1}{p_M^*} = \frac{1}{nrM}.$$

Consider the cube functional

$$a(Q) = [w]_{A_{p,q}}^{\frac{1}{p}}\,\ell(Q)\,\|\nabla f\|_{L^{p,q}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)}.$$

By Lemma 2.24, we have $a \in SD_{p_M^*}^{nM'}(w)$ with

$$\|a\| \le [w]_{A_r}^{1/(nrM)}.$$

Since $p \ge nr$, it follows that

$$p_M^* \ge rnM'.$$

Therefore, the second part of Theorem 2.5 applies and yields (2.32), with a constant depending on n, p and q. □

For pairs of weights, we can also obtain the corresponding counterpart of Proposition 2.23.

Proposition 2.30. *Let $(u, v) \in A_p$ and $u \in A_r$ with $1 \le r \le p$. Assume that $p \ge nr$. Then there exists a constant $c_{n,p} > 0$ such that, for every cube Q,*

$$\|f - f_Q\|_{\exp L\left(Q, \frac{u(x)\,dx}{u(Q)}\right)} \le c_{n,p}[u]_{A_\infty}\,[u]_{A_r}^{\frac{1}{rn}}\,[u, v]_{A_p}^{\frac{1}{p}}\,\ell(Q)\left(\frac{1}{u(Q)}\int_Q |\nabla f(x)|^p\,v(x)\,dx\right)^{\frac{1}{p}}.$$

2.5 Applications to weighted fractional Poincaré–Sobolev inequalities

In this section, we establish weighted Poincaré–Sobolev inequalities whose right-hand side is given by the local fractional Sobolev seminorm, also called the Gagliardo seminorm.

In order to consider two-weight A_p conditions, we need the following result adapted from **[MPW24]**.

Theorem 2.31. *Let $0 < \delta < 1$, $1 \le p < \infty$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and let $(u, v) \in A_p$. Then there exists a dimensional constant $c_n > 0$ such that*

$$\frac{1}{|Q|}\int_Q |f - f_Q| \le c_n [u,v]_{A_p}^{\frac{1}{p}} (1-\delta)^{\frac{1}{p}} \ell(Q)^{\delta} \left(\frac{1}{u(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, v(x)\, dx\right)^{\frac{1}{p}}, \quad (2.33)$$

for every cube $Q \subset \mathbb{R}^n$.

Proof. Assume first that $0 < \delta \le \frac{1}{2}$. By Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f - f_Q|\, dx &\le \frac{1}{|Q|^2}\int_Q\int_Q |f(x)-f(y)|\, dy\, dx \\
&\le C_n \ell(Q)^{\delta} \frac{1}{|Q|}\int_Q \left(\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\right)^{\frac{1}{p}} dx \\
&\le C_n [u,v]_{A_p}^{\frac{1}{p}} \ell(Q)^{\delta} \left(\frac{1}{u(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, v(x)\, dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $(1-\delta)^{\frac{1}{p}} \ge 2^{-\frac{1}{p}}$ in this range, this gives the claimed bound.

It remains to consider the case $\frac{1}{2} \le \delta < 1$. We follow the proof of the one-weight version in **[MPW24, Corollary 5.2]**. Fix a cube $Q \subset \mathbb{R}^n$ and let $0 \le \varepsilon \le \delta$. By (1.19) applied with parameter $\delta - \varepsilon$, we have

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\, dx \le C_n(1-\delta+\varepsilon)\ell(Q)^{\delta-\varepsilon}\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta-\varepsilon}}\, dy\, dx.$$

If $p = 1$, the conclusion follows from this estimate with $\varepsilon = 0$ and the definition of the two-weight A_1 condition. We may therefore assume that $p > 1$. For $0 < \varepsilon \le \delta$, Hölder's inequality in the y variable gives, for every $x \in Q$,

$$\int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta-\varepsilon}}\, dy \le C_n \frac{\ell(Q)^{\varepsilon}}{\varepsilon^{\frac{1}{p'}}}\left(\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\right)^{\frac{1}{p}}.$$

Hence, applying Hölder's inequality again, we obtain

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f - f_Q| &\le C_n \ell(Q)^{\delta}\frac{1-\delta+\varepsilon}{\varepsilon^{\frac{1}{p'}}}\frac{1}{|Q|}\int_Q \left(\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\right)^{\frac{1}{p}} dx \\
&\le C_n \ell(Q)^{\delta}\frac{1-\delta+\varepsilon}{\varepsilon^{\frac{1}{p'}}}\left(\frac{1}{|Q|}\int_Q v^{1-p'}\right)^{\frac{1}{p'}}\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, v(x)\, dx\right)^{\frac{1}{p}} \\
&\le C_n [u,v]_{A_p}^{\frac{1}{p}} \ell(Q)^{\delta}\frac{1-\delta+\varepsilon}{\varepsilon^{\frac{1}{p'}}}\left(\frac{1}{u(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, v(x)\, dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Choosing $\varepsilon = 1 - \delta$ yields $\frac{1-\delta+\varepsilon}{\varepsilon^{\frac{1}{p'}}} = 2(1-\delta)^{\frac{1}{p}}$. This proves the desired estimate also for $\frac{1}{2} \le \delta < 1$. □

Remark 2.32. In fact, the previous result is a small improvement of **[MPW24, Corollary 5.2]**, where an extra factor $\delta^{-1/p'}$ appears on the right-hand side. The argument of considering the cases $\delta \in (0, \frac{1}{2})$ and $\delta \in [\frac{1}{2}, 1)$ separately to eliminate the factor $\delta^{-1/p'}$ can be found in the one-weight setting in **[Gar25, p. 71]**.

We will use the following result from **[CMPR23, Lemma 6.2]**. Although it is stated in the product space setting, the argument for cubes is identical; therefore, we state it below in that form.

Lemma 2.33. *Let $w \in A_r$ with $1 \le r \le p$, let $0 < \delta < 1$ and let a be defined by*

$$a(Q) = \ell(Q)^{\delta} \left(\frac{1}{w(Q)} \int_Q \int_Q g(x,y)\, dy\, dx \right)^{\frac{1}{p}}. \tag{2.34}$$

Let $1 \le M \le \infty$ and let $p^*_{r,M,\delta}$ be defined by the condition

$$\frac{1}{p} - \frac{1}{p^*_{r,M,\delta}} = \frac{\delta}{nrM}. \tag{2.35}$$

Then:

1. *If $M > 1$, then $a \in SD^{\frac{nM'}{\delta}}_{p^*_{r,M,\delta}}(w)$ with $\|a\| \le [w]^{\frac{\delta}{nrM}}_{A_r}$.*
2. *If $M = 1$, then $a \in D_{p^*_{r,1,\delta}}(w)$ with $\|a\| \le [w]^{\frac{\delta}{nr}}_{A_r}$.*

We now prove the subcritical fractional Poincaré–Sobolev estimate. The proof follows the same pattern as in the first-order case: the weighted Poincaré inequality gives the starting estimate, the previous lemma gives the SD condition, and the truncation method upgrades the weak inequality to the corresponding Lorentz norm.

Theorem 2.34. *Let $0 < \delta < 1$. Let $w \in A_r$ with $1 \le r \le p$ and $1 \le p < \frac{nr}{\delta}$. We define the weighted fractional Sobolev exponent $p^*_{\delta,r}$ by the relation*

$$\frac{1}{p} - \frac{1}{p^*_{\delta,r}} = \frac{\delta}{n}\frac{1}{r}. \tag{2.36}$$

Then, there exists $c_n > 0$ such that, for all cubes Q,

$$\begin{aligned} \left\| f - f_Q \right\|_{L^{p^*_{\delta,r},p}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \le & c_n p^*_{\delta,r} [w]_{A_\infty} [w]^{\frac{1}{p}}_{A_p} [w]^{\frac{\delta}{rn}}_{A_r} (1-\delta)^{\frac{1}{p}} \cdot \\ & \cdot \ell(Q)^{\delta} \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, w(x)\, dx \right)^{\frac{1}{p}}. \end{aligned} \tag{2.37}$$

Proof. We start with the weighted fractional Poincaré estimate (2.33). Since $1 \le r \le p$, we have $A_r \subset A_p$, and hence (2.33) applies. Fix a cube Q and define

$$a(Q) := c_n (1-\delta)^{\frac{1}{p}} [w]^{\frac{1}{p}}_{A_p} \ell(Q)^{\delta} \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, w(x)\, dx \right)^{\frac{1}{p}}.$$

Then (2.33) gives

$$\frac{1}{|Q|} \int_Q |f - f_Q|\, dx \le a(Q),$$

for every cube Q. Since $a(Q)$ has the form (2.34), Lemma 2.33 implies that, for each $1 < M \le \infty$, the functional a verifies

$$a \in SD^{nM'/\delta}_{p^*_{r,M,\delta}}(w) \qquad \text{with} \qquad \|a\| \le [w]_{A_r}^{\delta/(nrM)}.$$

For $M = 1$ it satisfies the corresponding $D_{p^*_{r,1,\delta}}(w)$ condition with

$$\|a\| \le [w]_{A_r}^{\delta/(nr)}.$$

Here the exponent $p^*_{r,M,\delta}$ is defined by (2.35). Applying Theorem 2.5, or equivalently Theorem 2.2 in the case $M = 1$, we obtain

$$\|f - f_Q\|_{L^{p^*_{\delta,r},\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le c_n p^*_{\delta,r} [w]_{A_\infty} [w]_{A_r}^{\frac{\delta}{rn}} a(Q),$$

for every cube Q. Finally, applying the truncation method for fractional seminorms, Theorem 1.39, with $q = p^*_{\delta,r}$,

$$d\mu(x) = \frac{w(x)\,dx}{w(Q)}$$

and

$$K(y,z) = \frac{\ell(Q)^{\delta p}}{w(Q)} \frac{w(y)}{|y-z|^{n+\delta p}},$$

we obtain (2.37). □

Remark 2.35. In the subcritical range $1 \le p < \frac{nr}{\delta}$, the exponent $p^*_{\delta,r}$ is the natural fractional analogue of the Sobolev exponent. This exponent is sharp in a certain range of p and r (see Proposition 2.12 above and Remark 2.37 below).

We next consider the critical and supercritical range. In this case the fractional Sobolev exponent is no longer finite, and the conclusion is replaced by exponential integrability.

Theorem 2.36. *Let $0 < \delta < 1$. Let $w \in A_r$ with $1 \le r \le p$ and $p \ge \frac{nr}{\delta}$. Then there exists $c_{n,p} > 0$ such that, for every cube Q,*

$$\begin{aligned}\left\|f - f_Q\right\|_{\exp L\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le & c_{n,p} [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{\delta}{rn}} (1-\delta)^{\frac{1}{p}} \cdot \\ & \cdot \ell(Q)^{\delta} \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, w(x)\, dx\right)^{\frac{1}{p}}. \end{aligned} \tag{2.38}$$

Proof. We use the same functional

$$a(Q) := c_n (1-\delta)^{\frac{1}{p}} [w]_{A_p}^{\frac{1}{p}} \ell(Q)^{\delta} \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, w(x)\, dx\right)^{\frac{1}{p}}.$$

Again, (2.33) gives

$$\frac{1}{|Q|} \int_Q |f - f_Q|\, dx \le a(Q),$$

for every cube Q. Taking $M = \infty$ in Lemma 2.33, we get

$$a \in SD^{\frac{n}{\delta}}_p(w).$$

Since $p \geq \frac{nr}{\delta}$, we are in the critical/supercritical regime of Theorem 2.5. Therefore, the second part of Theorem 2.5 gives

$$\|f - f_Q\|_{\exp L\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \leq c_{n,p}[w]_{A_\infty}[w]_{A_r}^{\frac{1}{p}} a(Q).$$

Finally, since $p \geq \frac{nr}{\delta}$, we have $\frac{1}{p} \leq \frac{\delta}{nr}$, and hence

$$[w]_{A_r}^{\frac{1}{p}} \leq [w]_{A_r}^{\frac{\delta}{nr}}.$$

This yields (2.38). □

Remark 2.37. Using the counterexample constructed in Proposition 2.12, we see that Theorem 2.34 is optimal within the range satisfying both conditions $1 \leq r \leq p$ and $\frac{\delta}{n}p < r \leq 1 + \frac{\delta}{n}p$. We omit the details.

We conclude this section with the analogue of the previous results for pairs of weights.

Theorem 2.38. *Let $0 < \delta < 1$, $(u, v) \in A_p$, and $u \in A_r$ with $1 \leq r \leq p$.*

- *Assume that $1 \leq p < \frac{nr}{\delta}$. Then there exists $c_n > 0$ such that, for every cube Q,*

$$\left\|f - f_Q\right\|_{L^{p^*_{\delta,r},p}\left(Q, \frac{u(x)dx}{u(Q)}\right)} \leq c_n p^*_{\delta,r}[u]_{A_\infty}[u]_{A_r}^{\frac{\delta}{nr}}[u,v]_{A_p}^{\frac{1}{p}}(1-\delta)^{\frac{1}{p}} \cdot \\ \cdot \ell(Q)^\delta \left(\frac{1}{u(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}dy\, v(x)dx\right)^{\frac{1}{p}},$$

  where the weighted fractional Sobolev exponent $p^*_{\delta,r}$ is defined in (2.36)*.*

- *Assume that $p \geq \frac{nr}{\delta}$. Then there exists $c_{n,p} > 0$ such that, for every cube Q,*

$$\left\|f - f_Q\right\|_{\exp L\left(Q, \frac{u(x)dx}{u(Q)}\right)} \leq c_{n,p}[u]_{A_\infty}[u]_{A_r}^{\frac{\delta}{nr}}[u,v]_{A_p}^{\frac{1}{p}}(1-\delta)^{\frac{1}{p}} \cdot \\ \cdot \ell(Q)^\delta \left(\frac{1}{u(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}dy\, v(x)dx\right)^{\frac{1}{p}}.$$

Proof of Theorem 2.38. We start with the subcritical estimate (2.33). Fix a cube Q and define

$$a(Q) := c_n(1-\delta)^{\frac{1}{p}}[u,v]_{A_p}^{\frac{1}{p}}\ell(Q)^\delta\left(\frac{1}{u(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\, v(x)\,dx\right)^{\frac{1}{p}}.$$

Then (2.33) gives

$$\frac{1}{|Q|}\int_Q |f - f_Q|\,dx \leq a(Q),$$

for every cube Q. Since $a(Q)$ has the form (2.34), Lemma 2.33 implies that, for each $1 < M \leq \infty$, the functional a verifies

$$a \in SD^{nM'/\delta}_{p^*_{r,M,\delta}}(u) \qquad \text{with} \qquad \|a\| \leq [u]_{A_r}^{\delta/(nrM)}.$$

For $M = 1$ it satisfies the corresponding $D_{p^*_{r,1,\delta}}(u)$ condition with

$$\|a\| \leq [u]_{A_r}^{\delta/(nr)}.$$

Here the exponent $p^*_{r,M,\delta}$ is defined by (2.35). Applying Theorem 2.5, or equivalently Theorem 2.2 in the case $M = 1$, we obtain

$$\|f - f_Q\|_{L^{p^*_{r,\delta},\infty}\left(Q,\frac{u(x)\,dx}{u(Q)}\right)} \leq c_n p^*_{r,\delta}[u]_{A_\infty}[u]^{\frac{\delta}{nr}}_{A_r}\, a(Q),$$

for every cube Q. Finally, applying the truncation method for fractional seminorms, Theorem 1.39, with $q = p^*_{r,\delta}$,

$$d\mu(x) = \frac{u(x)\,dx}{u(Q)}$$

and

$$K(y,z) = \frac{\ell(Q)^{\delta p}}{u(Q)}\frac{v(y)}{|y-z|^{n+\delta p}},$$

we obtain the desired result.

The endpoint estimate (2.38) follows directly from Theorem 2.5 in the critical/supercritical regime, together with the fact that $a \in SD^{n/\delta}_p(u)$. This completes the proof. □

2.6 Weak weighted Poincaré–Sobolev inequalities and applications to A_1 weights

In this section, we investigate the weighted Poincaré–Sobolev inequality of type

$$\|f - f_Q\|_{Y_Q} \lesssim \ell(Q)\|\nabla f\|_{X_Q}, \quad \forall Q \subset \mathbb{R}^n,$$

where X_Q and Y_Q are Lorentz quasi-norms on Q induced by the measure $\frac{w(x)\,dx}{w(Q)}$. We call this a weighted $X - Y$ Poincaré–Sobolev inequality. In Section 2.4, we first proved the weighted $L^p - L^{p^*_r,\infty}$ Poincaré–Sobolev inequality (2.29) and then improved it to the weighted $L^p - L^{p^*_r,p}$ Poincaré–Sobolev inequality (2.30) by means of a truncation argument. Here we improve (2.29) from another perspective, replacing the right-hand side term with a smaller Lorentz norm. One result in this direction has already been shown in Theorem 2.26. In the present section we show that, if $w \in A_1$, the right-hand side term can be replaced by the weak norm $L^{p,\infty}$. More precisely, we prove a weighted $L^{p,\infty} - L^{p^*,\infty}$ Poincaré–Sobolev inequality for every $w \in A_1$.

The starting point is a weak Fefferman–Stein-type isoperimetric inequality.

Theorem 2.39. *Let $n \geq 2$ and let μ be a non-negative Borel measure. Then, there exists a dimensional constant $c_n > 0$ such that for all Lipschitz functions f we have*

$$\left\|f - f_Q\right\|_{L^{n',\infty}(Q,\mu)} \leq c_n \int_Q |\nabla f(x)|\mathcal{M}^c(\chi_Q\mu)(x)^{\frac{1}{n'}}\,dx, \tag{2.39}$$

for each cube Q.

The previous result was first established in **[FPW00]** in a different and more general setting. It was later considered in the Euclidean setting and extended in **[PR19]**, while a different proof, avoiding Minkowski's inequality, was given in **[PR23]**.

We first prove a Fefferman–Stein-type Poincaré–Sobolev inequality via an alternative approach, independent of the self-improving theory developed above. The estimate has weak norms on both sides and the maximal function of the weight on the right-hand side. We do not impose any assumptions on the weight.

Theorem 2.40. *Let w be a weight in $\mathbb{R}^n$ with $n \geq 2$ and let $1 < p < n$. Suppose f is a Lipschitz function. Then, there exists a constant $c_{n,p} > 0$ such that*

$$\left\|f - f_Q\right\|_{L^{p^*,\infty}(Q,w)} \leq c_{n,p} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}(Q,w)}, \tag{2.40}$$

for each cube Q.

Proof. By the definition of the Lorentz quasi-norm, it suffices to prove that there exists $c_{n,p} > 0$ such that

$$\left\|f - f_Q\right\|_{L^{p^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \leq c_{n,p} w(Q)^{\frac{1}{n}} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}. \tag{2.41}$$

Indeed, this is equivalent to (2.40), since $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. For simplicity, set

$$A := w(Q)^{\frac{1}{n}} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}.$$

If $A = 0$, the desired estimate is trivial. Hence, we may assume that $A > 0$. Let $t_0 = C_{n,p}A$, where $C_{n,p} > 0$ is a sufficiently large constant to be fixed below. Since

$$\left\|f - f_Q\right\|_{L^{p^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} = \max\left\{ \sup_{0<t\leq t_0} t\left(\frac{w(\{x \in Q : |f(x) - f_Q| > t\})}{w(Q)}\right)^{\frac{1}{p^*}}, \sup_{t_0 \leq t} t\left(\frac{w(\{x \in Q : |f(x) - f_Q| > t\})}{w(Q)}\right)^{\frac{1}{p^*}} \right\},$$

we consider two cases.

Case 1:

$$\sup_{0<t\leq t_0} t\left(\frac{w(\{x \in Q : |f(x) - f_Q| > t\})}{w(Q)}\right)^{\frac{1}{p^*}} \geq \sup_{t_0 \leq t} t\left(\frac{w(\{x \in Q : |f(x) - f_Q| > t\})}{w(Q)}\right)^{\frac{1}{p^*}}.$$

In this case, we have

$$\left\|f - f_Q\right\|_{L^{p^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \leq t_0 = C_{n,p}A,$$

and the desired estimate follows.

Case 2:

$$\sup_{0<t\leq t_0} t\left(\frac{w(\{x \in Q : |f(x) - f_Q| > t\})}{w(Q)}\right)^{\frac{1}{p^*}} \leq \sup_{t_0 \leq t} t\left(\frac{w(\{x \in Q : |f(x) - f_Q| > t\})}{w(Q)}\right)^{\frac{1}{p^*}}.$$

If the last supremum is zero, then the desired estimate is immediate. Otherwise, by the definition of the supremum, there exists $t_1 \geq t_0$ such that

$$\begin{aligned}\frac{t_1}{2}\left(\frac{w\left(\left\{x \in Q : |f(x) - f_Q| > \frac{t_1}{2}\right\}\right)}{w(Q)}\right)^{\frac{1}{p^*}} &\leq \left\|f - f_Q\right\|_{L^{p^*,\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \\ &= \sup_{t_0 \leq t} t\left(\frac{w(\{x \in Q : |f(x) - f_Q| > t\})}{w(Q)}\right)^{\frac{1}{p^*}} \\ &\leq 2t_1\left(\frac{w\left(\left\{x \in Q : |f(x) - f_Q| > t_1\right\}\right)}{w(Q)}\right)^{\frac{1}{p^*}}.\end{aligned}$$

This implies that

$$w\left(\left\{x \in Q : |f(x) - f_Q| > \frac{t_1}{2}\right\}\right) \leq c_{n,p}\, w\left(\left\{x \in Q : |f(x) - f_Q| > t_1\right\}\right). \tag{2.42}$$

Now, let us focus on the set

$$E_{t_1} := \{x \in Q : |f(x) - f_Q| > t_1\}.$$

To further study it, we introduce the truncation operator τ_λ defined as follows:

$$\tau_\lambda(g)(x) := \begin{cases} 0, & \text{if } g(x) \leq \lambda, \\ g(x) - \lambda, & \text{if } \lambda < g(x) \leq 2\lambda, \\ \lambda, & \text{if } g(x) > 2\lambda. \end{cases}$$

Note that, for each $x \in E_{t_1}$, we have

$$\frac{t_1}{2} = \tau_{\frac{t_1}{2}}(|f - f_Q|)(x) \leq \left|\tau_{\frac{t_1}{2}}(|f - f_Q|)(x) - \left(\tau_{\frac{t_1}{2}}(|f - f_Q|)\right)_Q\right| + \left(\tau_{\frac{t_1}{2}}(|f - f_Q|)\right)_Q.$$

Using the fact that $\tau_\lambda(g)(x) \leq g(x)$, Kolmogorov's inequality, and Theorem 2.39 with the Lebesgue measure, we obtain

$$\begin{aligned}\left(\tau_{\frac{t_1}{2}}(|f - f_Q|)\right)_Q &\leq \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \\ &\leq c_n \left\|f - f_Q\right\|_{L^{n',\infty}\left(Q, \frac{dx}{|Q|}\right)} \\ &\leq c_n \frac{1}{|Q|^{\frac{1}{n'}}}\int_Q |\nabla f(x)|\,dx.\end{aligned}$$

We now insert the weight in the previous estimate. Since

$$\frac{w(Q)}{|Q|} \leq \inf_{x \in Q} \mathcal{M}(w\chi_Q)(x),$$

and since $\mathcal{M}$ and $\mathcal{M}^c$ are pointwise comparable, we have

$$\begin{aligned}
\left(\tau_{\frac{t_1}{2}}(|f-f_Q|)\right)_Q &\le c_n \frac{1}{w(Q)^{\frac{1}{n'}}} \inf_{x\in Q} \mathcal{M}(w\chi_Q)(x)^{\frac{1}{n'}} \int_Q |\nabla f(x)|\,dx \\
&\le c_n \frac{1}{w(Q)^{\frac{1}{n'}}} \int_Q |\nabla f(x)| \mathcal{M}^c(w\chi_Q)(x)^{\frac{1}{n'}}\,dx \\
&= c_n w(Q)^{\frac{1}{n}} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^1\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \\
&\le c_{n,p} w(Q)^{\frac{1}{n}} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \\
&= c_{n,p} A.
\end{aligned}$$

In the last inequality, we used Kolmogorov's inequality again. Choosing the constant $C_{n,p} = 4c_{n,p}$ in the definition of t_0 sufficiently large, and recalling that $t_1 \ge t_0$, we obtain

$$\left(\tau_{\frac{t_1}{2}}(|f-f_Q|)\right)_Q \le \frac{t_1}{4}.$$

As a result,

$$E_{t_1} \subset \left\{ x \in Q : \left| \tau_{\frac{t_1}{2}}(|f-f_Q|)(x) - \left(\tau_{\frac{t_1}{2}}(|f-f_Q|)\right)_Q \right| \ge \frac{t_1}{4} \right\}.$$

Applying Theorem 2.39 to $\tau_{\frac{t_1}{2}}(|f-f_Q|)$ with the measure $w(x)\,dx$, we get

$$\begin{aligned}
\frac{t_1}{4} w(E_{t_1})^{\frac{1}{n'}} &\le \frac{t_1}{4} w\left(\left\{ x \in Q : \left| \tau_{\frac{t_1}{2}}(|f-f_Q|)(x) - \left(\tau_{\frac{t_1}{2}}(|f-f_Q|)\right)_Q \right| \ge \frac{t_1}{4} \right\} \right)^{\frac{1}{n'}} \\
&\le \left\| \tau_{\frac{t_1}{2}}(|f-f_Q|) - \left(\tau_{\frac{t_1}{2}}(|f-f_Q|)\right)_Q \right\|_{L^{n',\infty}(Q,w)} \\
&\le c_n \int_Q \left| \nabla\left(\tau_{\frac{t_1}{2}}(|f-f_Q|)\right)(x) \right| \mathcal{M}^c(w\chi_Q)(x)^{\frac{1}{n'}}\,dx \\
&\le c_n \int_{\left\{x\in Q : \frac{t_1}{2} < |f(x)-f_Q| < t_1\right\}} |\nabla f(x)| \frac{\mathcal{M}^c(w\chi_Q)(x)^{\frac{1}{n'}}}{w(x)} w(x)\,dx.
\end{aligned}$$

By Hölder's inequality with Lorentz norms (1.2), we have

$$\frac{t_1}{4} w(E_{t_1})^{\frac{1}{n'}} \le c_{n,p} \left\| \chi_{\left\{x\in Q : \frac{t_1}{2} < |f(x)-f_Q| < t_1\right\}} \right\|_{L^{p',1}(Q,w)} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}(Q,w)}.$$

Moreover, by (2.42),

$$\begin{aligned}
\left\| \chi_{\left\{x\in Q : \frac{t_1}{2} < |f(x)-f_Q| < t_1\right\}} \right\|_{L^{p',1}(Q,w)} &= c_p w\left(\left\{ x \in Q : \frac{t_1}{2} < |f(x)-f_Q| < t_1 \right\} \right)^{\frac{1}{p'}} \\
&\le c_{n,p} w\left(E_{t_1}\right)^{\frac{1}{p'}}.
\end{aligned}$$

Therefore,

$$t_1 w(E_{t_1})^{\frac{1}{n'}} \le c_{n,p} w(E_{t_1})^{\frac{1}{p'}} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}(Q,w)}.$$

Since $\frac{1}{n'} - \frac{1}{p'} = \frac{1}{p^*}$, we obtain

$$t_1 w(E_{t_1})^{\frac{1}{p^*}} \leq c_{n,p} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}(Q,w)}. \tag{2.43}$$

Combining everything together, we have

$$\begin{aligned}
\left\| f - f_Q \right\|_{L^{p^*,\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} &\leq 2t_1 \left(\frac{w(E_{t_1})}{w(Q)} \right)^{\frac{1}{p^*}} \\
&\leq c_{n,p} \left(\frac{1}{w(Q)} \right)^{\frac{1}{p^*}} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}(Q,w)} \\
&= c_{n,p} w(Q)^{\frac{1}{n}} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \\
&= c_{n,p} A.
\end{aligned}$$

This proves (2.41), and concludes the proof. □

The previous result is in the spirit of **[PR19, Theorem 1.21]**. In fact, it improves it in the range $p > 1$ in two ways. On the one hand, the right-hand side involves the weaker Lorentz quasi-norm $L^{p,\infty}$; on the other hand, the oscillation on the left-hand side is measured with respect to the unweighted average f_Q, rather than the weighted average $f_{Q,w}$. Moreover, by combining (2.40) with the truncation argument, we recover **[PR19, Theorem 1.21]** as a consequence.

We briefly indicate this last point. Using (2.40) and (1.3), we obtain

$$\begin{aligned}
\left\| f - f_Q \right\|_{L^{p^*,\infty}(Q,w)} &\leq c_{n,p} \left\| |\nabla f| \frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w} \right\|_{L^{p,\infty}(Q,w)} \\
&\leq c_{n,p} \left(\int_Q |\nabla f(x)|^p \frac{\mathcal{M}^c(w\chi_Q)(x)^{\frac{p}{n'}}}{w(x)^{p-1}}\, dx \right)^{\frac{1}{p}}.
\end{aligned}$$

Moreover, following the argument used in the proof of Theorem 2.40, starting from the classical L^1 Poincaré inequality on cubes and inserting the weight as in that proof, we also get

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx \leq c_{n,p} w(Q)^{-\frac{1}{p^*}} \left(\int_Q |\nabla f(x)|^p \frac{\mathcal{M}^c(w\chi_Q)(x)^{\frac{p}{n'}}}{w(x)^{p-1}}\, dx \right)^{\frac{1}{p}}.$$

We now apply Theorem 1.35 with $r = p \leq q = p^*$,

$$d\mu(x) = \frac{w(x)\,dx}{w(Q)}$$

and

$$d\nu(x) = w(Q)^{-\frac{p}{p^*}} \frac{\mathcal{M}^c(w\chi_Q)(x)^{\frac{p}{n'}}}{w(x)^{p-1}}\, dx.$$

Thus,

$$\left\| f - f_Q \right\|_{L^{p^*,p}(Q,\mu)} \leq c_{n,p} \left(\int_Q |\nabla f(x)|^p\, d\nu(x) \right)^{\frac{1}{p}}.$$

Multiplying by $w(Q)^{\frac{1}{p^*}}$, we conclude that

$$\begin{aligned}\left(\int_Q |f(x)-f_Q|^{p^*} w(x)\,dx\right)^{\frac{1}{p^*}} &\le \|f-f_Q\|_{L^{p^*,p}(Q,w)}\\ &\le c_{n,p}\left(\int_Q |\nabla f(x)|^p \frac{\mathcal{M}^c(w\chi_Q)(x)^{\frac{p}{n'}}}{w(x)^{p-1}}\,dx\right)^{\frac{1}{p}},\end{aligned}$$

which is a refinement of **[PR19, Theorem 1.21]**, since the unweighted average f_Q appears on the left-hand side.

For A_1 weights, we can derive the following weighted Poincaré–Sobolev inequality with weak norms.

Corollary 2.41. *Let $w \in A_1$ be a weight in $\mathbb{R}^n$ with $n \ge 2$ and let $1 < p < n$. Then there exists a constant $c_{n,p} > 0$ such that*

$$\|f-f_Q\|_{L^{p^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} \le c_{n,p}[w]_{A_1}\ell(Q)\,\|\nabla f\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}, \tag{2.44}$$

for each cube Q.

Proof. Dividing by $w(Q)^{\frac{1}{p^*}}$ in (2.40) and using the definition of the Sobolev exponent p^*, we obtain

$$\begin{aligned}\|f-f_Q\|_{L^{p^*,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)} &\le c_{n,p}w(Q)^{\frac{1}{n}}\left\||\nabla f|\frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w}\right\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}\\ &= c_{n,p}\ell(Q)\left(\frac{w(Q)}{|Q|}\right)^{\frac{1}{n}}\left\||\nabla f|\frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w}\right\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}\\ &\le c_{n,p}\ell(Q)\inf_{x\in Q}\left(\mathcal{M}(w\chi_Q)(x)\right)^{\frac{1}{n}}\left\||\nabla f|\frac{\mathcal{M}^c(w\chi_Q)^{\frac{1}{n'}}}{w}\right\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}\\ &\le c_{n,p}\ell(Q)\left\||\nabla f|\frac{\mathcal{M}^c(w\chi_Q)}{w}\right\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}\\ &\le c_{n,p}[w]_{A_1}\ell(Q)\,\|\nabla f\|_{L^{p,\infty}\left(Q,\frac{w(x)\,dx}{w(Q)}\right)}.\end{aligned}$$

In the third inequality we used the fact that $\mathcal{M}$ and $\mathcal{M}^c$ are pointwise equivalent, and in the last inequality we used that $w \in A_1$. □

The unweighted global version of (2.44) is known for the Lebesgue measure **[Pee66b]** (see also **[Mal03]** for a different approach).

Remark 2.42. Regarding the optimal power of the A_1 constant in (2.44), we will show that $\frac{1}{p}$ is a lower bound for (2.44) to hold. Indeed, if (2.44) were to hold with a smaller exponent $[w]_{A_1}^{\alpha}$, $\alpha < 1/p$, then, since $\|\nabla f\|_{L^{p,\infty}(Q,w)} \le \|\nabla f\|_{L^p(Q,w)}$, the same bound would follow with the strong L^p norm of the gradient on the right-hand side. Combining this with the truncation method would then yield an improvement of the sharp weighted Poincaré–Sobolev inequality obtained in **[Cla25a, Theorem 2.4]** (see Theorem 3.11 in the following chapter), and hence lead to a contradiction.

2.7 Generalized Poincaré inequalities with polynomials

Fix a cube Q and an integer $m \in \mathbb{N} \cup \{0\}$. We consider the space $\mathcal{P}_m(Q)$ of polynomials of degree at most m in n variables restricted to the cube Q. We denote by $P_Q^m f$ the projection of a function f onto the space $\mathcal{P}_m(Q)$. We have the following property: there exists a constant $c_{n,m} > 0$ such that

$$\left\| P_Q^m f \right\|_{L^\infty(Q)} \leq c_{n,m} \frac{1}{|Q|} \int_Q |f(x)| dx. \tag{2.45}$$

We refer to **[PR19, Section 8]** for a detailed explanation of this setting. The polynomial framework here is an extension of the average we discussed in previous sections, as $P_Q^0 f = f_Q$.

We state the main result of this section.

Theorem 2.43. *Let $1 \leq p < \infty$, $1 \leq r < \infty$, $1 \leq s < \infty$, and let a be a functional over cubes. Let $w \in A_r$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally integrable function satisfying*

$$\frac{1}{|Q|} \int_Q |f(x) - P_Q^m f(x)| \, dx \leq a(Q),$$

for every cube Q.

- *Assume that $a \in SD_p^s(w)$ and $p < rs$. Then there exists a constant $c_{n,m} > 0$ such that, for every cube Q,*

$$\left\| f - P_Q^m f \right\|_{L^{p^*_{r,s},\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \leq c_{n,m} p^*_{r,s} \, [w]_{A_\infty} \, [w]_{A_r}^{\frac{1}{rs}} \, \|a\| \, a(Q), \tag{2.46}$$

  where $p^*_{r,s}$ is defined by the relation

$$\frac{1}{p} - \frac{1}{p^*_{r,s}} = \frac{1}{r\,s}.$$

- *Assume that $a \in SD_p^s(w)$ and $p \geq rs$. Then there exists a constant $c_{n,m,p} > 0$ such that, for every cube Q,*

$$\|f - P_Q^m f\|_{\exp L\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \leq c_{n,m,p} \, [w]_{A_\infty} \, [w]_{A_r}^{\frac{1}{p}} \, \|a\| \, a(Q). \tag{2.47}$$

Sketch of the proof. We follow the proof of Theorem 2.5, replacing averages by the projections $P_Q^m f$. Fix a cube Q and set $\Omega_t := \{x \in Q : \mathcal{M}_Q(f - P_Q^m f)(x) > t\}$. Let $\{Q_j\}_j$ be the maximal dyadic subcubes forming Ω_t. Then either $\Omega_t = Q$, or

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f - P_Q^m f| \, dx \leq 2^n t.$$

Using $P_{Q_j}^m f - P_Q^m f = P_{Q_j}^m (f - P_Q^m f)$ and (2.45), we have $\|P_{Q_j}^m f - P_Q^m f\|_{L^\infty(Q_j)} \leq C_m 2^n t$, and hence for a suitable $\kappa = \kappa(n, m) > 1$,

$$w(\Omega_{\kappa t}) \leq \sum_j w(E_{Q_j}), \qquad E_{Q_j} := \{x \in Q_j : \mathcal{M}_{Q_j}(|f - P_{Q_j}^m f|)(x) > t\}.$$

Introduce the polynomial sharp maximal function

$$\mathcal{M}_Q^{\#,m} f(x) := \sup_{x \in R \in \mathcal{D}(Q)} \frac{1}{|R|} \int_R |f - P_R^m f|.$$

For $\gamma > 0$ split each E_{Q_j} into the regions where $\mathcal{M}_Q^{\#,m} f \leq \gamma t$ and where $\mathcal{M}_Q^{\#,m} f > \gamma t$. The polynomial good-lambda inequality (obtained from **[CP21, Theorem 7.1]**) yields

$$w(\Omega_{\kappa t}) \leq c_1 e^{-\frac{c_2}{[w]_{A_\infty} \gamma}} w(\Omega_t) + w(\{x \in Q : \mathcal{M}_Q^{\#,m} f(x) > \gamma t\}).$$

The second term is estimated exactly as in the proof of (2.5) using the hypothesis $a \in SD_p^s(w)$. The exponential estimate (2.47) follows from (2.46) via Proposition 2.8. □

As a consequence of this result, we have the following Poincaré–Sobolev inequality for high-order derivatives.

Corollary 2.44. *Let $1 \leq m < n$. Let $w \in A_r$ with $1 \leq r \leq p$, $1 < p < \frac{n}{m} r$ and consider $p^*_{m,r}$ defined by*

$$\frac{1}{p} - \frac{1}{p^*_{m,r}} = \frac{m}{n} \frac{1}{r}.$$

Then, there exists a constant $c_{n,m} > 0$ such that

$$\|f - P_Q^{m-1} f\|_{L^{p^*_{m,r},\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \leq c_{n,m} p^*_{m,r} [w]_{A_\infty} [w]^{\frac{1}{p}}_{A_{p,p^*_{m,r}}} [w]^{\frac{m}{rn}}_{A_r} \ell(Q)^m \|\nabla^m f\|_{L^{p,p^*_{m,r}}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)},$$

for every cube Q.

The proof of this result is essentially the same as that of Theorem 2.26, but using the following starting inequality from **[Boj88]**: there exists a constant $C > 1$ such that

$$\frac{1}{|Q|} \int_Q |f(x) - P_Q^{m-1} f(x)|dx \leq C \ell(Q)^m \frac{1}{|Q|} \int_Q |\nabla^m f(x)|dx,$$

for every cube Q. The rest of the proof has already been shown in the previous section, we omit it here.

2.8 Self-improving of vector-valued Poincaré inequalities

In this section, we extend the main self-improving result of this chapter to the vector-valued setting, where the oscillation is measured through the ℓ_q-norm.

Theorem 2.45. *Let $q \geq 1$, $1 \leq p < \infty$, $1 \leq r < \infty$, $1 \leq s < \infty$, and let a be a functional over cubes. Let $w \in A_r$ and let $f : \mathbb{R}^n \longrightarrow \ell_q$ be a locally integrable function satisfying*

$$\frac{1}{|Q|} \int_Q \left\| f(x) - f_Q \right\|_{\ell_q} dx \leq a(Q),$$

for every cube Q.

- *Assume that $a \in SD_p^s(w)$ and $p < rs$. Then there exists a dimensional constant $c_n > 0$ such that, for every cube Q,*

$$\left\| \left\| f - f_Q \right\|_{\ell_q} \right\|_{L^{p^*_{r,s},\infty}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \leq c_n p^*_{r,s}\, [w]_{A_\infty}\, [w]_{A_r}^{\frac{1}{rs}}\, \|a\|\, a(Q),$$

where $p^*_{r,s}$ is defined by the relation

$$\frac{1}{p} - \frac{1}{p^*_{r,s}} = \frac{1}{r\,s}.$$

- *Assume that $a \in SD_p^s(w)$ and $p \geq rs$. Then there exists a constant $c_{n,p} > 0$ such that, for every cube Q,*

$$\left\| \left\| f - f_Q \right\|_{\ell_q} \right\|_{\exp L\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \leq c_{n,p}\, [w]_{A_\infty}\, [w]_{A_r}^{\frac{1}{p}}\, \|a\|\, a(Q).$$

Sketch of the proof. This is a direct vector-valued adaptation of the scalar proof: we apply the whole argument to the scalar function $x \mapsto \|f(x) - f_Q\|_{\ell_q}$. The Calderón–Zygmund decomposition and the key estimate $w(\Omega_{ct}) \leq \sum_j w(E_{Q_j})$ follow from the triangle inequality in ℓ_q and Minkowski's inequality. The good-lambda step applies to the corresponding sharp maximal function built from $\|f - f_R\|_{\ell_q}$, and the remaining estimate uses $a \in SD_p^s(w)$ exactly as in the scalar case. The exponential endpoint follows from the weak bounds via Proposition 2.8. □

We also remark that we can prove a vector-valued version of **[CP21, Theorem 1.1]**. Such a vector-valued extension would yield the corresponding good-lambda inequality needed in the previous argument.

In order to derive concrete consequences of this vector-valued self-improving result, we shall use the following theorem from **[MP23]**.

Theorem 2.46. *Let $q \geq 1$. There is a dimensional constant $c_n > 0$ such that, for any cube Q of $\mathbb{R}^n$ and for any vector-valued function $f : \mathbb{R}^n \longrightarrow \ell_q$ with components in $C^1(Q)$, the representation formula*

$$\left\| f(x) - f_Q \right\|_{\ell_q} \leq c_n I_1(\|\nabla f\|_{\ell_q}\, \chi_Q)(x)$$

holds for every $x \in Q$.

Since I_1 is self-adjoint, the previous result implies the following Poincaré inequality.

Theorem 2.47. *Let $q \geq 1$. There is a dimensional constant $c_n > 0$ such that, for any cube Q of $\mathbb{R}^n$ and for any vector-valued function $f : \mathbb{R}^n \longrightarrow \ell_q$ with components in $C^1(Q)$, we have*

$$\frac{1}{|Q|} \int_Q \left\| f(x) - f_Q \right\|_{\ell_q} dx \leq c_n \ell(Q) \frac{1}{|Q|} \int_Q \|\nabla f(x)\|_{\ell_q}\, dx.$$

Consequently, noting the fact that the truncation method is not known to work in the vector-valued setting (as noted in **[MP23, p. 2]**), we obtain the following analogue of Theorem 2.26.

Corollary 2.48. *Let $q \geq 1$. Let $w \in A_r$ with $1 \leq r \leq p$, $1 < p < nr$ and consider p_r^* defined by*

$$\frac{1}{p} - \frac{1}{p_r^*} = \frac{1}{n}\frac{1}{r}.$$

Then, there exists a dimensional constant $c_n > 0$ such that

$$\left\| \|f - f_Q\|_{\ell_q} \right\|_{L^{p_r^*,\infty}\left(Q, \frac{w\,dx}{w(Q)}\right)} \leq c_n p_r^* [w]_{A_\infty} [w]_{A_{p,p_r^*}}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{rn}} \ell(Q) \left\| \|\nabla f\|_{\ell_q} \right\|_{L^{p,p_r^*}\left(Q, \frac{w\,dx}{w(Q)}\right)},$$

for every cube Q.

2.9 Further extensions: Rectangles

In this section, we generalize Theorems 2.5 and 2.43 to the context of rectangles in $\mathbb{R}^n$. We borrow the terminology from **[CMPR23]**. We denote by $\mathcal{R}$ the family of rectangles in $\mathbb{R}^n$. By a rectangle, we mean the product of n intervals in $\mathbb{R}$. We will denote by $SD^s_{p,\mathcal{R}}(w)$ the condition analogous to the $SD^s_p(w)$ condition, considering rectangles instead of cubes. We will state the result in the context of polynomial approximation, with the same definitions as in the previous section.

Theorem 2.49. *Let $1 \leq p < \infty$, $1 \leq r < \infty$, $1 \leq s < \infty$, and let a be a functional over rectangles. Let $w \in A_{r,\mathcal{R}}$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally integrable function satisfying*

$$\frac{1}{|R|} \int_R |f(x) - P_R^m f(x)|\, dx \leq a(R),$$

for every rectangle R.

- *Assume that $a \in SD^s_{p,\mathcal{R}}(w)$ and $p < rs$. Then there exists a constant $c_{n,m} > 0$ such that, for every rectangle R,*

$$\left\| f - P_R^m f \right\|_{L^{p^*_{r,s},\infty}\left(R, \frac{w(x)\,dx}{w(R)}\right)} \leq c_{n,m} p^*_{r,s} [w]_{A_{\infty,\mathcal{R}}} [w]_{A_{r,\mathcal{R}}}^{\frac{1}{rs}} \|a\|\, a(R),$$

  where $p^*_{r,s}$ is defined by the relation

$$\frac{1}{p} - \frac{1}{p^*_{r,s}} = \frac{1}{r\,s}.$$

- *Assume that $a \in SD^s_{p,\mathcal{R}}(w)$ and $p \geq rs$. Then there exists a constant $c_{n,m,p} > 0$ such that, for every rectangle R,*

$$\|f - P_R^m f\|_{\exp L\left(R, \frac{w(x)\,dx}{w(R)}\right)} \leq c_{n,m,p} [w]_{A_{\infty,\mathcal{R}}} [w]_{A_{r,\mathcal{R}}}^{\frac{1}{p}} \|a\|\, a(R).$$

The proof is the same as the proof of Theorem 2.43 but using the Rising Sun Lemma from **[KLS05]** instead of Calderón–Zygmund decomposition. We state here the lemma for the sake of completeness.

Lemma 2.50. *Let R be a rectangle in $\mathbb{R}^n$ and let f be a function such that*

$$\frac{1}{|R|} \int_R f(x)dx \leq A.$$

Then, there exists a pairwise disjoint family of rectangles $\{R_j\}_j$ such that

- $R_j \subset R$ *for all* j.
- $\frac{1}{|R_j|}\int_{R_j} f(x)dx = A$ *for all* j.
- $f(x) \leq A$ *a.e.* $x \in R \setminus \bigcup_j R_j$.

Remark 2.51. The preceding lemma does not introduce any dimensional constant. The dimensional dependence in the previous result comes instead from the polynomial projection estimate (2.45). In the non-polynomial case, that is, when the oscillation is measured with respect to the average f_Q, Theorem 2.49 does not require any dimensional constants.

Chapter 3

Degenerate Poincaré–Sobolev inequalities via fractional integration

The results in this chapter are contained in the following works.

[Cla25a] Claros, A. *Degenerate Poincaré-Sobolev inequalities via fractional integration*, J. Funct. Anal., **289** (2025), no. 6, Paper No. 111000, DOI: 10.1016/j.jfa.2025.111000.

[CR26] Claros, A. and Rela, E. *Asymptotically sharp embedding of A_∞ into A_p for flat weights and applications to Poincaré-Sobolev inequalities*, arXiv preprint arXiv:2604.25873 (2026).

In this chapter, we first prove local weighted estimates for the Riesz potential, which are then used to establish weighted Poincaré–Sobolev inequalities with sharp dependence on the A_p constants. We then determine the optimal Sobolev exponent associated with a given weight w. The definition of this critical exponent p_w^* motivates the study of the embedding of A_∞ into A_p, which was carried out during my three-month research stay at the Universidad de Buenos Aires, hosted by Ezequiel Rela. As a main consequence of this embedding, we finally obtain a new Poincaré–Sobolev inequality for A_∞ weights with small A_∞ constant.

Although the results in **[Cla25a]** are formulated in terms of balls, their proofs adapt to cubes with only minor modifications. We have chosen to present this chapter in the setting of cubes in order to make this point explicit and, at the same time, maintain a coherent framework throughout the thesis.

3.1 Introduction

Weighted Poincaré–Sobolev inequalities play a key role in proving the local Hölder regularity of weak solutions of degenerate elliptic partial differential equations. In the celebrated paper **[FKS82]**, Fabes, Kenig, and Serapioni considered the following degenerate elliptic PDE, namely the operator

$$Lu = \operatorname{div}(A(x)\nabla u),$$

where A is an $n \times n$ symmetric matrix in $\mathbb{R}^n$ satisfying the following degenerate ellipticity condition: there exist $C_1, C_2 > 0$ such that

$$C_1 w(x)|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq C_2 w(x)|\xi|^2,$$

where w is a weight in the A_2 Muckenhoupt class. The Moser iteration method is a classical and powerful technique to prove the local Hölder regularity of the weak solutions of a given elliptic partial differential equation (see **[HKM06]**). The method has two important steps: the $(2, 2)$ Poincaré inequality and the $(2^*, 2)$ Poincaré–Sobolev inequality, where $2^* = \frac{2n}{n-2} > 2$ is the classical Sobolev exponent. In order to apply the Moser iteration method to the degenerate setting, in **[FKS82]** weighted Poincaré and Poincaré–Sobolev inequalities are proved, namely

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^q w(x)dx\right)^{\frac{1}{q}} \leq C_w \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}}, \tag{3.1}$$

for all cubes Q, $1 \leq p \leq q < \infty$ and w in some A_p class of Muckenhoupt. This type of inequality has attracted the attention of many authors **[CF85; CW85; CW92; FPW98; Hur98; PR19; PR23; SW92]**.

As discussed in Chapter 2, inequalities of the form (3.1) can be obtained through the self-improving theory of generalized Poincaré inequalities. This approach was introduced in **[FPW98]**, refined in **[MP98; LP05]**, and later improved quantitatively in **[CP21]**. Its main feature is that suitable discrete summability conditions on the functional appearing on the right-hand side of a generalized Poincaré inequality imply stronger integrability estimates.

In this direction, Pérez and Rela **[PR19]** obtained weighted Poincaré–Sobolev inequalities improving those in **[FKS82]**, and studied both the quantitative dependence of the constant C_w in (3.1) and the precise admissible range of the exponent q in terms of n, p, and w. In particular, **[PR19, Corollary 1.22]** proves (3.1) for all $w \in A_1$, with $q = \frac{np}{n-p}$ and $C_w = c_{n,p}[w]_{A_1}$. They also show that the exponent of $[w]_{A_1}$ cannot be smaller than $\frac{1}{p}$, and conjecture that the same inequality holds with C_w replaced by $c_{n,p}[w]_{A_1}^{1/p}$ (see **[PR19, Conjecture 1.23]**).

In this chapter we follow a complementary route, based on fractional integration. This perspective is particularly useful for tracking the sharp dependence on the Muckenhoupt constants. As a consequence of our main result, we answer this conjecture positively (see Theorem 3.11 and the following remark).

The classical approach for obtaining Poincaré–Sobolev inequalities is through the use of a subrepresentation formula, namely, there exists $c_n > 0$ such that for each cube Q,

$$|f(x) - f_Q| \leq c_n I_1(|\nabla f|\chi_Q)(x), \tag{3.2}$$

for every $x \in Q$ (see Theorem 1.28). As we previously discussed in Section 1.4.2, the mapping properties of I_α are well known. Indeed, the classical technique consists in taking appropriate norms on both sides of (3.2) and then applying the boundedness properties of the Riesz potential to readily deduce the corresponding Poincaré–Sobolev estimates. In **[MW74]**, Muckenhoupt and Wheeden studied global weighted bounds for the fractional integral operator.

Consequently, the problem of deriving weighted Poincaré–Sobolev inequalities is naturally reduced to establishing suitable bounds for the Riesz potential. In the present chapter, we are interested in

studying local bounds of I_α with A_p weights. This framework for obtaining weighted Poincaré–Sobolev inequalities was first used in **[CF85]**. This type of inequality was studied in **[ACS09]**. We state the main result of that paper.

Theorem 3.1. *Let $n \geq 2$, $\alpha \in (0, n)$ and $1 < p < \frac{n}{\alpha}$. Let $w \in A_p$ and given $r \geq 1$ consider q defined by the relation*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}\frac{1}{r}.$$

Then, there exist positive constants $k = k(p, n)$ and $C = C(\alpha, p, n)$ such that if

$$p - \frac{k}{[w]_{A_p}^{\frac{1}{p-1}}} < r \leq p, \tag{3.3}$$

then

$$\left(\frac{1}{w(Q)}\int_Q |I_\alpha f(x)|^q w(x)dx\right)^{\frac{1}{q}} \leq C[w]_{A_p}^{\frac{nr-\alpha}{nr(p-1)}} \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |f(x)|^p w(x)dx\right)^{\frac{1}{p}}, \tag{3.4}$$

for any cube $Q \subset \mathbb{R}^n$ of side length $\ell(Q)$ and every function $f \in L^p(Q, w)$ (continued by 0 outside Q). Moreover, the exponent $\frac{nr-\alpha}{nr(p-1)}$ of the A_p constant is sharp, in the sense that the statement is false if $\frac{nr-\alpha}{nr(p-1)}$ is replaced by any smaller exponent.

Our main result, Theorem 3.3, improves Theorem 3.1 in several ways. First, we improve the r's obtained in (3.3), which produces a larger q. Second, we obtain a better quantitative bound in (3.4), in terms of a mixed $A_p - A_\infty$ constant (see Theorem 1.13). We also extend the result to the case $p = 1$, replacing the L^q average on the left-hand side with a weak-L^q average.

We can derive other types of inequalities using different subrepresentation formulas and our main results. For instance, we can obtain Poincaré–Sobolev inequalities for high-order derivatives. In addition, we obtain weighted fractional Poincaré–Sobolev inequalities, namely,

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^q w(x)dx\right)^{\frac{1}{q}} \leq C_w \ell(Q)^\delta \left(\frac{1}{w(Q)}\int_Q\int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}}dy\, w(x)dx\right)^{\frac{1}{p}},$$

where the object on the right-hand side is the local weighted Gagliardo seminorm of f, $1 \leq p \leq q < \infty$ and $0 < \delta < 1$. We can prove this inequality with the interesting BBM extra gain factor $(1-\delta)^{\frac{1}{p}}$ in the inequality (see Theorem 3.15).

We conclude by answering the problem of finding the optimal exponent q in (3.1) for the class A_p. Specifically, for a fixed $1 \leq p < n$ and a fixed $w \in A_p$, we define an exponent p_w^* and prove that the inequality (3.1) cannot hold for exponents $q > p_w^*$ (see Definition 3.21 and Theorem 3.24).

3.2 Weighted Hedberg's inequality

To establish our main results concerning the local bounds for the Riesz potential, we first need a suitable pointwise estimate. The classical unweighted Hedberg inequality (see Theorem 1.25) bounds the fractional integral by a combination of the Hardy–Littlewood maximal function and the global L^p

norm of the function. In our context, we require a localized, weighted version of this estimate. The following lemma provides exactly such a bound, carefully tracking the dependence on the weight and the constants.

Lemma 3.2. *Let $n \geq 1$, $\alpha \in (0, n)$, let $1 < p < \frac{n}{\alpha}$ and let w be a weight. Let $1 < r \leq p$ and consider q defined by the relation*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}\frac{1}{r}.$$

Then, there exists a dimensional constant $c_n > 0$ such that for any cube Q such that $0 < \int_Q w(y)^{1-r'} dy < \infty$ and every function $f \in L^p(Q, w)$, we have

$$|I_\alpha f(x)| \leq \frac{c_n}{\alpha} p_\alpha^* \mathcal{M} f(x)^{\frac{p}{q}} \|f\|_{L^p(Q,w)}^{1-\frac{p}{q}} \left(\int_Q w(y)^{1-r'} dy \right)^{\frac{\alpha}{nr'}},$$

for any $x \in Q$.

This inequality was first proved in **[CF85]** in the case $\alpha = 1$ and $w \in A_p$. The proof of this inequality for $w \in A_p$ is essentially contained in the proof of Theorem 1.1 from **[ACS09]**; we have chosen to include the proof because we need to be precise with the main parameters involved.

Proof. Let Q be a cube such that $0 < \int_Q w(y)^{1-r'} dy < \infty$, and let $f \in L^p(Q, w)$. Fix $x \in Q$ and let $\delta > 0$. We can decompose as follows,

$$|I_\alpha f(x)| \leq \int_{|x-y| \leq \delta} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy + \int_{|x-y| > \delta} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy = I + II.$$

We will estimate each term separately. Firstly,

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \int_{\frac{\delta}{2^{k+1}} < |x-y| \leq \frac{\delta}{2^k}} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \\ &\leq \sum_{k=0}^{\infty} \left(\frac{2^{k+1}}{\delta} \right)^{n-\alpha} \int_{|x-y| \leq \frac{\delta}{2^k}} |f(y)| dy \\ &\leq 2^{n-\alpha} \omega_n \delta^\alpha \mathcal{M} f(x) \sum_{k=0}^{\infty} \frac{1}{2^{\alpha k}} \\ &= \frac{2^n \omega_n}{2^\alpha - 1} \delta^\alpha \mathcal{M} f(x), \end{aligned}$$

where ω_n is the Lebesgue measure of the unit ball. Secondly, using Hölder's inequality, we obtain

$$\begin{aligned} II &= \int_{|x-y| > \delta} \frac{|f(y)|}{|x-y|^{n-\alpha}} w(y)^{\frac{1}{p}} w(y)^{-\frac{1}{p}} dy \\ &\leq \|f\|_{L^p(Q,w)} \left(\int_{Q \cap \{|x-y| > \delta\}} \frac{w(y)^{1-p'}}{|x-y|^{(n-\alpha)p'}} dy \right)^{\frac{1}{p'}}. \end{aligned}$$

Let $1 < r < p$, we use Hölder's inequality again with $\frac{1-r'}{1-p'} > 1$,

$$\begin{aligned}\int_{Q\cap\{|x-y|>\delta\}} \frac{w(y)^{1-p'}}{|x-y|^{(n-\alpha)p'}}dy &\le \left(\int_Q w(y)^{1-r'}dy\right)^{\frac{1-p'}{1-r'}} \\ &\quad\cdot\left(\int_{Q\cap\{|x-y|>\delta\}} \frac{dy}{|x-y|^{(n-\alpha)p'\left(\frac{1-r'}{1-p'}\right)'}}\right)^{\frac{1}{\left(\frac{1-r'}{1-p'}\right)'}} \\ &= \left(\int_Q w(y)^{1-r'}dy\right)^{\frac{r-1}{p-1}} \left(\int_{Q\cap\{|x-y|>\delta\}} \frac{dy}{|x-y|^{\frac{p(n-\alpha)}{p-r}}}\right)^{\frac{p-r}{p-1}}.\end{aligned}$$

Hence,

$$II \le \|f\|_{L^p(Q,w)} \left(\int_Q w(y)^{1-r'}dy\right)^{\frac{r-1}{p}} \left(\int_{Q\cap\{|x-y|>\delta\}} \frac{dy}{|x-y|^{\frac{p(n-\alpha)}{p-r}}}\right)^{\frac{p-r}{p}}.$$

We can estimate the second integral,

$$\begin{aligned}\left(\int_{Q\cap\{|x-y|>\delta\}} \frac{dy}{|x-y|^{\frac{p(n-\alpha)}{p-r}}}\right)^{\frac{p-r}{p}} &= \left(\int_0^\infty \left|\left\{y\in Q : \delta < |x-y| < t^{-\frac{p-r}{p(n-\alpha)}}\right\}\right| dt\right)^{\frac{p-r}{p}} \\ &= \left(\int_0^{\delta^{-\frac{p(n-\alpha)}{p-r}}} \left|\left\{y\in Q : \delta < |x-y| < t^{-\frac{p-r}{p(n-\alpha)}}\right\}\right| dt\right)^{\frac{p-r}{p}} \\ &\le \left(\int_0^{\delta^{-\frac{p(n-\alpha)}{p-r}}} \left|\left\{y\in Q : |x-y| < t^{-\frac{p-r}{p(n-\alpha)}}\right\}\right| dt\right)^{\frac{p-r}{p}} \\ &\le \left(\int_0^{\delta^{-\frac{p(n-\alpha)}{p-r}}} \left|B\left(x, t^{-\frac{p-r}{p(n-\alpha)}}\right)\right| dt\right)^{\frac{p-r}{p}} \\ &= \left(\omega_n \int_0^{\delta^{-\frac{p(n-\alpha)}{p-r}}} t^{-\frac{(p-r)n}{p(n-\alpha)}} dt\right)^{\frac{p-r}{p}} \\ &\le c_n \frac{p(n-\alpha)}{n-\alpha p}\delta^{\alpha-\frac{nr}{p}}.\end{aligned}$$

If $r = p$, we obtain a similar estimate,

$$II \le \|f\|_{L^p(Q,w)}\, \delta^{\alpha-n} \left(\int_Q w(y)^{1-p'}dy\right)^{\frac{1}{p'}}.$$

Therefore,

$$|I_\alpha f(x)| \le C\left(\delta^\alpha \mathcal{M}f(x) + \delta^{\alpha-\frac{nr}{p}}\, \|f\|_{L^p(Q,w)} \left(\int_Q w(y)^{1-r'}dy\right)^{\frac{r-1}{p}}\right),$$

where $C = c_n \max\left\{\frac{1}{2^\alpha-1}, \frac{p(n-\alpha)}{n-\alpha p}\right\}$. Observe that since $\frac{\log 2}{2^\alpha-1} \le \frac{1}{\alpha}$, we have

$$C \le c_n p\left(\frac{1}{\alpha} + \frac{n-\alpha}{n-\alpha p}\right) \le \frac{c_n p}{\alpha(n-\alpha p)} = \frac{c_n}{\alpha} p^*_\alpha.$$

Without loss of generality, we may assume that $\mathcal{M}f(x) > 0$, since otherwise $f = 0$ and there is nothing to prove. The choice

$$\delta = \left(\frac{\|f\|_{L^p(Q,w)} \left(\int_Q w(y)^{1-r'} dy \right)^{\frac{r-1}{p}}}{\mathcal{M}f(x)} \right)^{\frac{p}{nr}}$$

yields the desired inequality

$$|I_\alpha f(x)| \leq \frac{c_n}{\alpha} p_\alpha^* \mathcal{M}f(x)^{\frac{p}{q}} \|f\|_{L^p(Q,w)}^{1-\frac{p}{q}} \left(\int_Q w(y)^{1-r'} dy \right)^{\frac{\alpha}{nr'}},$$

for each $x \in Q$. □

3.3 Weighted local bounds for fractional integrals with A_p weights

We begin this section by noting that the A_p condition is necessary for (3.4). Let us assume that for a weight w we have the inequality

$$\left(\frac{1}{w(Q)} \int_Q |I_\alpha f(x)|^q w(x) dx \right)^{\frac{1}{q}} \leq C\ell(Q)^\alpha \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x) dx \right)^{\frac{1}{p}}, \tag{3.5}$$

for fixed p, q, for any cube Q and every $f \in L^p(Q, w)$, then necessarily $w \in A_p$. Observe that for any cube Q and any non-negative function f, we have

$$\frac{\ell(Q)^\alpha}{|Q|} \int_Q f(y) dy \leq \mathcal{M}_\alpha f(x) \leq c_n I_\alpha f(x),$$

for all $x \in Q$. Then, applying (3.5) we have,

$$\frac{\ell(Q)^\alpha}{|Q|} \int_Q f(y) dy \leq C\ell(Q)^\alpha \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x) dx \right)^{\frac{1}{p}},$$

and this is equivalent to $w \in A_p$ (as we have seen in (1.8)). We have proved that if (3.5) holds for a weight w, then $w \in A_p$. This observation shows that it is necessary to consider weights in class A_p for this type of inequality.

Our first result is the following theorem, which is an improvement of the main result of **[ACS09]**, Theorem 3.1 above.

Theorem 3.3. *Let $n \geq 1$, $\alpha \in (0, n)$ and $1 < p < \frac{n}{\alpha}$. Let $w \in A_r$ with $1 \leq r \leq p$. Consider q_r defined by the relation*

$$\frac{1}{p} - \frac{1}{q_r} = \frac{\alpha}{n} \frac{\tau_n [\sigma]_{A_\infty}}{1 + r(\tau_n [\sigma]_{A_\infty} - 1)}, \tag{3.6}$$

where $\sigma = w^{1-r'}$ and $\tau_n = 2^{n+1}$ if $r > 1$ and $q_1 = p_\alpha^* = \frac{np}{n - \alpha p}$. Then, there exists a dimensional constant $c_n > 0$ such that, if $r > 1$ we have

$$\left(\frac{1}{w(Q)} \int_Q |I_\alpha f(x)|^{q_r} w(x) dx \right)^{\frac{1}{q_r}} \leq \frac{c_n}{\alpha} p_\alpha^* (p')^{\frac{1}{q_r}} [w]_{A_r}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{q_r}} \ell(Q)^\alpha \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x) dx \right)^{\frac{1}{p}},$$

for any cube Q and every function $f \in L^p(Q,w)$, and if $r = 1$ we have,

$$\left(\frac{1}{w(Q)}\int_Q |I_\alpha f(x)|^{q_1} w(x)dx\right)^{\frac{1}{q_1}} \leq \frac{c_n}{\alpha} p_\alpha^* \, (p')^{\frac{1}{q_1}} \, [w]_{A_1}^{\frac{1}{p}} \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |f(x)|^p w(x)dx\right)^{\frac{1}{p}},$$

for any cube Q and every function $f \in L^p(Q,w)$.

The proof relies on the pointwise estimate provided by the weighted Hedberg inequality (Lemma 3.2 above). To obtain the strong norm inequality, we simply combine this pointwise control of the fractional integral operator I_α with the following weighted bound for the Hardy–Littlewood maximal operator $\mathcal{M}$,

$$\|\mathcal{M}f\|_{L^p(w)} \leq c_n \left(\frac{p}{p-r}\right)^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{p}} \|f\|_{L^p(w)},$$

where $w \in A_r$ with $1 \leq r < p$ (see (1.13) for more details). This approach, together with the left-openness property of the class A_r, naturally produces a mixed-type constant, which includes the $[\sigma]_{A_\infty}^{1/q_r}$ term in the estimate.

As recalled in Theorem 1.26, the good-lambda method of Muckenhoupt and Wheeden **[MW74]** gives a comparison between I_α and $\mathcal{M}_\alpha$ for A_∞ weights. The quantitative form discussed there, which follows from the exponential good-lambda inequality of Honzík and Jaye **[HJ12]**, yields a linear dependence on the Fujii–Wilson constant $[w]_{A_\infty}$. Although this approach is effective, and the fractional maximal operator $\mathcal{M}_\alpha$ is often easier to handle, it would introduce an additional A_∞ factor into our estimates. Since we want to be precise with the dependence on constants, we instead work directly with the weighted Hedberg inequality above.

Remark 3.4. Note that in the case $r = p$, the exponent q_p defined by (3.6) is larger than the range of q's obtained in Theorem 3.1. We remark the improvement of the A_p constant, using basic properties of A_p weights (see Section 1.3.2) we have,

$$[w]_{A_p}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{q}} \leq c_n [w]_{A_p}^{\frac{1}{p}} [\sigma]_{A_{p'}}^{\frac{1}{q}} = c_n [w]_{A_p}^{\frac{1}{p}} [w]_{A_p}^{\frac{p'-1}{q}} = c_n [w]_{A_p}^{\frac{nr-\alpha}{nr(p-1)}}.$$

This observation shows that the exponent $\frac{1}{p}$ in $[w]_{A_r}$ and the exponent $\frac{1}{q_r}$ in $[\sigma]_{A_\infty}$ cannot be improved in general. Specifically, if $r = p$, any improvement of these exponents would imply an improved exponent of the A_p constant in Theorem 3.1, contradicting its sharpness **[ACS09, p. 1269]**.

We first prove a result, from which the theorem above will be deduced. To state it, it is convenient to introduce the following definition of the set of p for which a given weight w belongs to the class A_p.

Definition 3.5. Let $w \in A_\infty$, we define

$$I(w) := \{p \geq 1 : w \in A_p\} = \begin{cases} [1,\infty) & \text{if } w \in A_1 \\ (\ell_w,\infty) & \text{if } w \in A_\infty \setminus A_1 \end{cases},$$

where

$$\ell_w := \inf I(w).$$

We note here that if $w \in A_\infty$ then $w \in A_p$ for each $p > e^{c_n [w]_{A_\infty}}$ (see **[HP16, Theorem 1.3]**). Therefore, for each weight $w \in A_\infty$, the quantity $e^{c_n [w]_{A_\infty}}$ is an upper bound of ℓ_w.

Remark 3.6. Let $w \in A_\infty$, there is a significant difference between $w \in A_1$ and $w \in A_\infty \setminus A_1$. In the first case, $w \in A_{\ell_w} = A_1$. On the other hand, if $w \in A_\infty \setminus A_1$, then $w \notin A_{\ell_w}$. In other words, the infimum ℓ_w is a minimum if and only if $w \in A_1$.

We will first prove the following result, which implies Theorem 3.3.

Proposition 3.7. *Let $n \geq 1$, $\alpha \in (0, n)$ and $1 < p < \frac{n}{\alpha}$. Let $w \in A_p$ and consider q_r defined by the relation*

$$\frac{1}{p} - \frac{1}{q_r} = \frac{\alpha}{n}\frac{1}{r}, \tag{3.7}$$

where $1 \leq r < p$. If $r \in I(w)$ (that is, if $w \in A_r$), then there exists a dimensional constant $c_n > 0$ such that

$$\left(\frac{1}{w(Q)}\int_Q |I_\alpha f|^{q_r} w\right)^{\frac{1}{q_r}} \leq \frac{c_n}{\alpha} p_\alpha^* \left(\frac{p}{p-r}\right)^{\frac{1}{q_r}} [w]_{A_r}^{\frac{1}{p}} \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |f|^p w\right)^{\frac{1}{p}}, \tag{3.8}$$

for any cube Q and every function $f \in L^p(Q, w)$.

Proof. Let $r \in I(w)$. Let us assume first that $1 < r < p$, then we have $0 < \int_Q w(y)^{1-r'} dy < \infty$, since $w \in A_r$. Applying Lemma 3.2 we obtain

$$|I_\alpha f(x)| \leq \frac{c_n}{\alpha} p_\alpha^* \mathcal{M} f(x)^{\frac{p}{q_r}} \|f\|_{L^p(Q,w)}^{1-\frac{p}{q_r}} \left(\int_Q w(y)^{1-r'} dy\right)^{\frac{\alpha}{nr'}},$$

for all $x \in Q$. Therefore,

$$\begin{aligned}
\left(\frac{1}{w(Q)}\int_Q |I_\alpha f|^{q_r} w\right)^{\frac{1}{q_r}} &\leq \frac{c_n}{\alpha} p_\alpha^* \|f\|_{L^p(Q,w)}^{1-\frac{p}{q_r}} \left(\int_Q w^{1-r'}\right)^{\frac{\alpha}{nr'}} \left(\frac{1}{w(Q)}\int_Q |\mathcal{M} f|^p w\right)^{\frac{1}{q_r}} \\
&= \frac{c_n}{\alpha} p_\alpha^* \|f\|_{L^p(Q,w)}^{1-\frac{p}{q_r}} \frac{w(Q)^{\frac{\alpha}{nr}}}{w(Q)^{\frac{1}{p}}} \left(\int_Q w^{1-r'}\right)^{\frac{\alpha(r-1)}{nr}} \left(\int_Q |\mathcal{M} f|^p w\right)^{\frac{1}{q_r}} \\
&\leq \frac{c_n}{\alpha} p_\alpha^* [w]_{A_r}^{\frac{\alpha}{nr}} \ell(Q)^\alpha \frac{1}{w(Q)^{\frac{1}{p}}} \|f\|_{L^p(Q,w)}^{1-\frac{p}{q_r}} \left(\int_{\mathbb{R}^n} |\mathcal{M} f|^p w\right)^{\frac{1}{q_r}},
\end{aligned}$$

where we have used the A_r condition. Since $w \in A_r$ with $r < p < \infty$, we use the estimate (1.13),

$$\begin{aligned}
\left(\frac{1}{w(Q)}\int_Q |I_\alpha f|^{q_r} w\right)^{\frac{1}{q_r}} &\leq \frac{c_n}{\alpha} p_\alpha^* \left(\frac{p}{p-r}\right)^{\frac{1}{q_r}} [w]_{A_r}^{\frac{\alpha}{nr}} \ell(Q)^\alpha \frac{1}{w(Q)^{\frac{1}{p}}} \|f\|_{L^p(Q,w)}^{1-\frac{p}{q_r}} [w]_{A_r}^{\frac{1}{q_r}} \|f\|_{L^p(Q,w)}^{\frac{p}{q_r}} \\
&= \frac{c_n}{\alpha} p_\alpha^* \left(\frac{p}{p-r}\right)^{\frac{1}{q_r}} [w]_{A_r}^{\frac{1}{p}} \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |f|^p w\right)^{\frac{1}{p}}.
\end{aligned}$$

This yields (3.8). Let us consider the remaining case $r = 1 \in I(w)$. We apply the classical Hedberg's inequality (see Theorem 1.25) and we use $w \in A_1$,

$$|I_\alpha f(x)| \leq \frac{c_n}{\alpha} p_\alpha^* \mathcal{M} f(x)^{\frac{p}{q_1}} \|f\|_{L^p(Q,dx)}^{1-\frac{p}{q_1}} \leq \frac{c_n}{\alpha} p_\alpha^* [w]_{A_1}^{\frac{\alpha}{n}} \ell(Q)^\alpha \mathcal{M} f(x)^{\frac{p}{q_1}} \|f\|_{L^p\left(Q, \frac{w(x)dx}{w(Q)}\right)}^{1-\frac{p}{q_1}}.$$

Using again (1.13) with $r = 1$, we have,

$$\begin{aligned}\|I_\alpha f\|_{L^{q_1}(Q,w)} \leq & \frac{c_n}{\alpha} p_\alpha^* [w]_{A_1}^{\frac{\alpha}{n}} \ell(Q)^\alpha \|f\|_{L^p\left(Q,\frac{w(x)dx}{w(Q)}\right)}^{1-\frac{p}{q_1}} \|\mathcal{M}f\|_{L^p(Q,w)}^{\frac{p}{q_1}} \\ \leq & \frac{c_n}{\alpha} p_\alpha^* (p')^{\frac{1}{q_1}} [w]_{A_1}^{\frac{\alpha}{n}} \ell(Q)^\alpha \|f\|_{L^p\left(Q,\frac{w(x)dx}{w(Q)}\right)}^{1-\frac{p}{q_1}} [w]_{A_1}^{\frac{1}{q_1}} \|f\|_{L^p(Q,w)}^{\frac{p}{q_1}} \\ = & \frac{c_n}{\alpha} p_\alpha^* (p')^{\frac{1}{q_1}} [w]_{A_1}^{\frac{1}{p}} \ell(Q)^\alpha w(Q)^{\frac{1}{q_1}} \|f\|_{L^p\left(Q,\frac{w(x)dx}{w(Q)}\right)},\end{aligned}$$

and this yields the desired inequality (3.8) which completes the proof. □

Remark 3.8. It should be noted that a similar inequality holds for pairs of weights $(u, v) \in A_r$ with $1 \leq r < p$ (we refer to Definition 1.23 for the precise definition). The proof follows the same argument, using a version of Lemma 1.12 for pairs of weights $(u, v) \in A_r$.

Using the precise left-openness property (Corollary 1.15) and the previous proposition, we are able to prove Theorem 3.3.

Proof of Theorem 3.3. We may assume $r > 1$ since the case $r = 1$ has been proved in the previous proposition. Using Corollary 1.15, since $w \in A_r$, then $w \in A_{r-\varepsilon}$ where

$$\varepsilon = \frac{r-1}{\tau_n [\sigma]_{A_\infty}},$$

and $[w]_{A_{r-\varepsilon}} \leq 2^{r-1}[w]_{A_r}$. Observe that

$$r - \varepsilon = r - \frac{r-1}{\tau_n [\sigma]_{A_\infty}} = \frac{1 + r(\tau_n [\sigma]_{A_\infty} - 1)}{\tau_n [\sigma]_{A_\infty}}.$$

Since $w \in A_{r-\varepsilon}$, then $r - \varepsilon \in I(w)$ and we can apply Proposition 3.7,

$$\begin{aligned}\left(\frac{1}{w(Q)}\int_Q |I_\alpha f|^{q_r} w\right)^{\frac{1}{q_r}} \leq & \frac{c_n}{\alpha} p_\alpha^* \left(\frac{p}{p-(r-\varepsilon)}\right)^{\frac{1}{q_r}} [w]_{A_{r-\varepsilon}}^{\frac{1}{p}} \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |f|^p w\right)^{\frac{1}{p}} \\ \leq & \frac{c_n}{\alpha} p_\alpha^* 2^{\frac{1}{p'}} \left(\frac{p}{p-r+\varepsilon}\right)^{\frac{1}{q_r}} [w]_{A_r}^{\frac{1}{p}} \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |f|^p w\right)^{\frac{1}{p}} \\ \leq & \frac{c_n}{\alpha} p_\alpha^* \left(\frac{p}{p-1}\tau_n [\sigma]_{A_\infty}\right)^{\frac{1}{q_r}} [w]_{A_r}^{\frac{1}{p}} \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |f|^p w\right)^{\frac{1}{p}} \\ = & \frac{c_n}{\alpha} p_\alpha^* (p')^{\frac{1}{q_r}} [w]_{A_r}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{q_r}} \ell(Q)^\alpha \left(\frac{1}{w(Q)}\int_Q |f|^p w\right)^{\frac{1}{p}},\end{aligned}$$

where we have used the bound

$$\begin{aligned}\frac{p}{p-r+\varepsilon} = & \frac{p\tau_n [\sigma]_{A_\infty}}{p\tau_n [\sigma]_{A_\infty} - 1 - r(\tau_n [\sigma]_{A_\infty} - 1)} \\ \leq & \frac{p\tau_n [\sigma]_{A_\infty}}{p\tau_n [\sigma]_{A_\infty} - 1 - p(\tau_n [\sigma]_{A_\infty} - 1)} \\ = & \frac{p}{p-1}\tau_n [\sigma]_{A_\infty},\end{aligned}$$

since $r \leq p$ and $\tau_n [\sigma]_{A_\infty} > 1$. This completes the proof. □

The following result is an analogous estimate to the previous theorem, but considering the weak norm on the left-hand side, this allows us to reach the case $p = 1$.

Theorem 3.9. *Let $n \geq 1$, $\alpha \in (0, n)$ and $1 \leq p < \frac{n}{\alpha}$. Let $w \in A_r$ with $1 \leq r \leq p$. Consider q_r defined in* (3.6) *if $r > 1$ and $q_1 = p^*_\alpha = \frac{np}{n-\alpha p}$. Then, there exists a dimensional constant $c_n > 0$ such that*

$$\|I_\alpha f\|_{L^{q_r,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq \frac{c_n}{\alpha} p^*_\alpha [w]^{\frac{1}{p}}_{A_r} \ell(Q)^\alpha \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x)dx\right)^{\frac{1}{p}},$$

for any cube Q and every function $f \in L^p(Q, w)$.

This result is interesting in itself because we obtain a better constant than that obtained with the strong norm, and in order to derive Poincaré–Sobolev inequalities, it is enough to prove weak weighted estimates.

As before, we need the following proposition, which implies Theorem 3.9.

Proposition 3.10. *Let $n \geq 1$, $\alpha \in (0, n)$ and $1 \leq p < \frac{n}{\alpha}$. Let $w \in A_p$ and consider q_r defined in* (3.7) *with $1 \leq r \leq p$. If $r \in I(w)$, then there exists a dimensional constant $c_n > 0$ such that*

$$\|I_\alpha f\|_{L^{q_r,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq \frac{c_n}{\alpha} p^*_\alpha [w]^{\frac{1}{p}}_{A_r} \ell(Q)^\alpha \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x)dx\right)^{\frac{1}{p}},$$

for any cube Q and every function $f \in L^p(Q, w)$.

The proof of the previous proposition is analogous to the proof of Proposition 3.7, using (1.11) instead of (1.13), which produces a better constant. We remark that if $p = 1$, we can use the classical Hedberg's inequality, and an analogous argument produces the desired inequality for each $w \in A_1$.

Proof of Theorem 3.9. We may assume $r > 1$ since the case $r = 1$ is contained in the previous proposition. Using the left-openness property stated in Corollary 1.15, since $w \in A_r$, then $w \in A_{r-\varepsilon}$ where

$$\varepsilon = \frac{r-1}{\tau_n [\sigma]_{A_\infty}},$$

and we have $[w]_{A_{r-\varepsilon}} \leq 2^{r-1}[w]_{A_r}$. Note that $q = q_{r-\varepsilon}$. Since $w \in A_{r-\varepsilon}$, then $r - \varepsilon \in I(w)$ and we can apply Proposition 3.10,

$$\begin{aligned}\|I_\alpha f\|_{L^{q,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} &\leq \frac{c_n}{\alpha} p^*_\alpha [w]^{\frac{1}{p}}_{A_{r-\varepsilon}} \ell(Q)^\alpha \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x)dx\right)^{\frac{1}{p}} \\ &\leq \frac{c_n}{\alpha} p^*_\alpha [w]^{\frac{1}{p}}_{A_r} \ell(Q)^\alpha \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x)dx\right)^{\frac{1}{p}}.\end{aligned}$$

□

3.4 Applications to Poincaré–Sobolev inequalities

Now, by combining the subrepresentation formula (3.2), the results from the previous section (specifically Theorem 3.9), and the truncation method stated in Theorem 1.35, we can prove the following result.

Theorem 3.11. *Let $n \geq 2$ and let $1 \leq p < n$. Let $w \in A_r$ for some $1 \leq r \leq p$ and consider q_r defined by the relation*

$$\frac{1}{p} - \frac{1}{q_r} = \frac{1}{n} \frac{\tau_n [\sigma]_{A_\infty}}{1 + r(\tau_n [\sigma]_{A_\infty} - 1)}, \tag{3.9}$$

where $\sigma = w^{1-r'}$ and $\tau_n = 2^{n+1}$ if $r > 1$ and $q_1 = p^* = \frac{np}{n-p}$. Then, there exists a dimensional constant $c_n > 0$ such that,

$$\left(\frac{1}{w(Q)} \int_Q |f(x) - f_Q|^{q_r} w(x) dx \right)^{\frac{1}{q_r}} \leq c_n p^* [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x) dx \right)^{\frac{1}{p}}, \tag{3.10}$$

for any cube Q, and the exponent $\frac{1}{p}$ on the A_r constant is sharp. Moreover, using the truncation method, we can self-improve the left-hand side to the Lorentz norm $L^{q_r,p}$,

$$\left\| f - f_Q \right\|_{L^{q_r,p}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq c_n p^* [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x) dx \right)^{\frac{1}{p}},$$

for any cube Q.

Remark 3.12. In the case $r = 1$, this theorem answers positively to the Conjecture 1.23 from **[PR19]**. Observe that $q_1 = p^* = \frac{np}{n-p}$ is the classical Sobolev exponent. In that case, we have proved that there exists a dimensional constant $c_n > 0$ such that

$$\left(\frac{1}{w(Q)} \int_Q |f(x) - f_Q|^{p^*} w(x) dx \right)^{\frac{1}{p^*}} \leq c_n p^* [w]_{A_1}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x) dx \right)^{\frac{1}{p}},$$

for all $w \in A_1$, and the exponent $\frac{1}{p}$ in the A_1 constant is sharp (cf. Proposition 7.4 **[PR19]**).

Remark 3.13. The new weighted Sobolev exponent q_r defined by

$$\frac{1}{p} - \frac{1}{q_r} = \frac{1}{n} \frac{\tau_n [\sigma]_{A_\infty}}{1 + r(\tau_n [\sigma]_{A_\infty} - 1)}$$

is, to the best of our knowledge, new to the literature. This exponent is larger than the best weighted Sobolev exponents obtained in **[FPW98]** and **[PR19]** for the A_r class of weights with $1 < r \leq p$, and also improves the exponents obtained in Chapter 2.

Proof of Theorem 3.11. Since $w \in A_r$ we have $r \in I(w)$, and we can apply Theorem 3.9 with $\alpha = 1$ to obtain

$$\begin{aligned} \left\| f - f_Q \right\|_{L^{q_r,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} &\leq c_n \left\| I_1(|\nabla f| \chi_Q) \right\|_{L^{q_r,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \\ &\leq c_n p^* [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x) dx \right)^{\frac{1}{p}}, \end{aligned}$$

where in the first inequality, we have used (3.2). Moreover, by (2.1), (1.8), and the inclusion $A_r \subset A_p$,

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq c_n [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x) \, dx \right)^{\frac{1}{p}}.$$

We obtain the desired inequalities using the truncation method stated in Theorem 1.35. The sharpness of the exponent $\frac{1}{p}$ on the A_r constant in (3.10) will be proved in Proposition 3.27 below. □

Remark 3.14. Using the quantitative open property established in Corollary 1.18, we can improve the Sobolev exponent q_r obtained in Theorem 3.11. Specifically, the weighted Poincaré–Sobolev inequality (3.10) holds with a larger exponent q defined by the relation

$$\frac{1}{p}-\frac{1}{q}=\frac{1}{n}\frac{1+\tau_n([\sigma]_{A_\infty}-1)}{1+r\tau_n([\sigma]_{A_\infty}-1)}.$$

However, this improvement comes at the cost of a worse constant. The bound on the right-hand side becomes $c_{n,p}[w]_{A_r}^{\frac{1}{p}}[\sigma]_{A_\infty}^{\frac{1}{p'}}$, meaning we lose the sharp dependence on $[w]_{A_r}^{\frac{1}{p}}$ achieved in Theorem 3.11. Indeed, in the case $r=p$, we obtain

$$[w]_{A_p}^{\frac{1}{p}}[\sigma]_{A_\infty}^{\frac{1}{p'}}\leq c_n[w]_{A_p}^{\frac{2}{p}}.$$

We now present our result on fractional Poincaré–Sobolev inequalities.

Theorem 3.15. *Let $n\geq 2$, $\delta\in(0,1)$, $1\leq p<\frac{n}{\delta}$ and let $w\in A_1$. Then, there exists a dimensional constant $c_n>0$ such that,*

$$\begin{aligned}\left(\frac{1}{w(Q)}\int_Q|f(x)-f_Q|^{p(\frac{n}{\delta})'}w(x)dx\right)^{\frac{1}{p(\frac{n}{\delta})'}}\leq& c_n\frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1+\frac{1}{p'}}}[w]_{A_1}^{\frac{1}{p}}\ell(Q)^{\delta}\\&\cdot\left(\frac{1}{w(Q)}\int_Q\int_Q\frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}dy\,w(x)dx\right)^{\frac{1}{p}},\end{aligned}$$

for any cube Q.

Remark 3.16. The exponent $\frac{1}{p}$ on the A_1 constant is better than that obtained using self-improving methods in **[HMPV23]**. In the case $p=1$, we also improve the quantitative dependence of the A_1 constant proved in Theorem 5.7 from **[MPW24]**. In addition, for weights with large A_1 constant, our range of local integrability is larger than that obtained in **[HMPV23]**. More precisely, $p\left(\frac{n}{\delta}\right)'>\frac{np(1+\log[w]_{A_1})}{n(1+\log[w]_{A_1})-\delta p}$ if and only if $[w]_{A_1}>e^{p-1}$.

Proof of Theorem 3.15. Fix a cube Q, using (1.19) and Lemma 1.31 with $\alpha=\delta$ and $r=p$, we obtain the following subrepresentation formula,

$$|f(x)-f_Q|\leq c_n\frac{(1-\delta)^{\frac{1}{p}}}{\delta^{\frac{1}{p'}}}\ell(Q)^{\frac{\delta}{p'}}I_\delta(g_{f,Q}^p\chi_Q)(x)^{\frac{1}{p}},\tag{3.11}$$

for almost every $x\in Q$, where

$$g_{f,Q}(x)=\left(\int_Q\frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}dy\right)^{\frac{1}{p}}.$$

Using (3.11) and Proposition 3.10 with $\alpha = \delta$, $p = r = 1$ and noting that $q_1 = \left(\frac{n}{\delta}\right)'$ we have,

$$\begin{aligned}
\left\|f - f_Q\right\|_{L^p\left(\frac{n}{\delta}\right)',\infty\left(Q,\frac{w\,dx}{w(Q)}\right)} \leq & c_n \frac{(1-\delta)^{\frac{1}{p}}}{\delta^{\frac{1}{p'}}} \ell(Q)^{\frac{\delta}{p'}} \left\|I_\delta(g_{f,Q}^p \chi_Q)^{\frac{1}{p}}\right\|_{L^p\left(\frac{n}{\delta}\right)',\infty\left(Q,\frac{w(x)dx}{w(Q)}\right)} \\
= & c_n \frac{(1-\delta)^{\frac{1}{p}}}{\delta^{\frac{1}{p'}}} \ell(Q)^{\frac{\delta}{p'}} \left\|I_\delta(g_{f,Q}^p \chi_Q)\right\|^{\frac{1}{p}}_{L\left(\frac{n}{\delta}\right)',\infty\left(Q,\frac{w(x)dx}{w(Q)}\right)} \\
\leq & \frac{c_n}{\delta(n-\delta)} \frac{(1-\delta)^{\frac{1}{p}}}{\delta^{\frac{1}{p'}}} \ell(Q)^{\frac{\delta}{p'}} [w]_{A_1}^{\frac{1}{p}} \ell(Q)^{\frac{\delta}{p}} \left(\frac{1}{w(Q)} \int_Q |g_{f,Q}|^p w\,dx\right)^{\frac{1}{p}} \\
\leq & c_n \frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1+\frac{1}{p'}}} [w]_{A_1}^{\frac{1}{p}} \ell(Q)^{\delta} \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}} dy\, w(x) dx\right)^{\frac{1}{p}}.
\end{aligned}$$

The conclusion of the theorem comes from Theorem 1.39. □

Remark 3.17. The method used above introduces an additional factor $\delta^{-1-1/p'}$ in the estimate. However, compared with the self-improving approach used in Theorem 2.34, it yields a better dependence on the A_1 constant.

We now present our results for higher-order derivatives. One important tool that we need is the following subrepresentation formula for high-order derivatives (we refer to **[BH93]** and also to the book **[AH96]**).

Theorem 3.18. *Let $1 \leq m < n$. There exists a constant $C = C(n,m) > 0$ such that, for every cube $Q \subset \mathbb{R}^n$ and every $f \in W^{m,1}(Q)$,*

$$|f(x) - P_Q^{m-1} f(x)| \leq C\, I_m\big(|\nabla^m f| \chi_Q\big)(x),$$

for almost every $x \in Q$.

Following the notations established in the previous chapter, specifically in Section 2.7, and combining the pointwise estimate stated in Theorem 3.18 with Theorem 3.9, we obtain the following weak Poincaré–Sobolev inequality for high-order derivatives.

Theorem 3.19. *Let $n \geq 2$, $m \geq 1$ and let $1 \leq p < \frac{n}{m}$. Let $w \in A_r$ for some $1 \leq r \leq p$ and consider q_r defined by the relation*

$$\frac{1}{p} - \frac{1}{q_r} = \frac{m}{n} \frac{\tau_n [\sigma]_{A_\infty}}{1 + r(\tau_n [\sigma]_{A_\infty} - 1)}, \tag{3.12}$$

where $\sigma = w^{1-r'}$ and $\tau_n = 2^{n+1}$ if $r > 1$ and $q_1 = p_m^* = \frac{np}{n-mp}$. Then, there exists a constant $C = C(n,m) > 0$ such that,

$$\left\|f - P_Q^{m-1} f\right\|_{L^{q_r,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq C p_m^* [w]_{A_r}^{\frac{1}{p}} \ell(Q)^m \left(\frac{1}{w(Q)} \int_Q |\nabla^m f(x)|^p w(x) dx\right)^{\frac{1}{p}},$$

for any cube Q.

Proof. Let Q be a cube. By the pointwise subrepresentation formula in Theorem 3.18, we have for every $x \in Q$,

$$|f(x) - P_Q^{m-1} f(x)| \leq C I_m(|\nabla^m f|\chi_Q)(x).$$

Taking the weighted weak $L^{q_r,\infty}$ quasi-norm over Q on both sides and applying Theorem 3.9 with $\alpha = m$, we deduce

$$\begin{aligned}\left\|f - P_Q^{m-1} f\right\|_{L^{q_r,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} &\leq C \left\|I_m(|\nabla^m f|\chi_Q)\right\|_{L^{q_r,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \\ &\leq C p_m^* [w]_{A_r}^{\frac{1}{p}} \ell(Q)^m \left(\frac{1}{w(Q)} \int_Q |\nabla^m f(x)|^p w(x) dx\right)^{\frac{1}{p}}.\end{aligned}$$

This concludes the proof. □

The truncation argument does not work for high-order derivatives (see **[Haj01, p. 121]**). In order to obtain strong weighted norm inequalities, we have to use Theorem 3.3, although it introduces the constant $[\sigma]_{A_\infty}^{1/q_r}$ in the estimates.

Theorem 3.20. *Let $n \geq 2$, $m \geq 1$ and let $1 < p < \frac{n}{m}$. Let $w \in A_r$ for some $1 \leq r \leq p$ and consider q_r defined in* (3.12) *if $r > 1$ and $q_1 = p_m^* = \frac{np}{n-mp}$. Then, there exists a constant $C = C(n,m) > 0$ such that, if $r > 1$ we have*

$$\left(\frac{1}{w(Q)} \int_Q |f - P_Q^{m-1} f|^{q_r} w\right)^{\frac{1}{q_r}} \leq C p_m^* [w]_{A_r}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{q_r}} \ell(Q)^m \left(\frac{1}{w(Q)} \int_Q |\nabla^m f|^p w\right)^{\frac{1}{p}},$$

for any cube Q, and if $r = 1$ we have

$$\left(\frac{1}{w(Q)} \int_Q |f - P_Q^{m-1} f|^{q_1} w\right)^{\frac{1}{q_1}} \leq C p_m^* [w]_{A_1}^{\frac{1}{p}} \ell(Q)^m \left(\frac{1}{w(Q)} \int_Q |\nabla^m f|^p w\right)^{\frac{1}{p}},$$

for any cube Q.

The proof of the previous theorem follows exactly the same argument as above, replacing Theorem 3.9 by Theorem 3.3.

3.5 Optimality of the weighted Sobolev exponent

In this section, we address the problem of determining the optimal Sobolev exponent associated with a given weight $w \in A_p$. To this end, we introduce in the next definition our candidate for this optimal exponent.

Definition 3.21. Let $w \in A_\infty$ and $1 \leq p < n\ell_w$. We define the weighted Sobolev exponent p_w^* by the relation

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n}\frac{1}{\ell_w}.$$

Remark 3.22. If $w \in A_1$, then $\ell_w = 1$, and hence $p_w^* = p^*$ is the classical Sobolev exponent.

We can now establish the following result for this exponent by using the local estimates for the Riesz potential I_1 obtained in the previous sections. More precisely, we prove that the corresponding Poincaré–Sobolev inequality holds for every exponent $1 \le q < p_w^*$.

Theorem 3.23. *Let $w \in A_\infty$, and let $1 \le p < n\ell_w$ be such that $w \in A_p$. Then, for each $1 \le q < p_w^*$, we have*

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^q w(x)dx\right)^{\frac{1}{q}} \le C_w \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}}, \tag{3.13}$$

for any cube Q, where C_w is of the form $c_n p^* [w]_{A_r}^{\frac{1}{p}}$ for some $r > \ell_w$ depending on q.

Proof of Theorem 3.23. Let $w \in A_\infty$, and let $1 \le p < n\ell_w$ be such that $w \in A_p$. We may assume without loss of generality that $q_p < q < p_w^*$, where q_p is defined by (3.7), since the remaining cases follow from this one by Jensen's inequality. Then, there exists $r \in (\ell_w, p)$ such that $q = q_r$, where

$$\frac{1}{p} - \frac{1}{q_r} = \frac{1}{n}\frac{1}{r}.$$

Since $\ell_w < r$, we have $w \in A_r$, and therefore, using (3.2) together with Proposition 3.10, we obtain

$$\begin{aligned}\left\|f - f_Q\right\|_{L^{q_r,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} &\le c_n \left\|I_1(|\nabla f|\chi_Q)\right\|_{L^{q_r,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \\ &\le c_n p^* [w]_{A_r}^{\frac{1}{p}} \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}},\end{aligned}$$

for every cube Q. Using the truncation argument, we obtain the desired inequality (3.13). □

The exponent p_w^* and this type of inequality in the range $p \le q < p_w^*$ are implicit in the proof of the weighted Poincaré–Sobolev inequalities in **[FKS82]**, as remarked in **[CMR18]**. The following result shows that the exponent p_w^* cannot be improved.

Theorem 3.24. *We can find a weight $w \in A_p$ such that if the inequality*

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^q w(x)dx\right)^{\frac{1}{q}} \le C_w \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}}$$

holds for every cube Q and for all Lipschitz functions f, then $q \le p_w^*$.

The previous result shows that the inequality (3.13) is false in general if $q > p_w^*$. On the other hand, we know that it is true in the case $q = p_w^*$ when $w \in A_1 \subset A_p$ (Theorem 3.11 with $r = 1$). We do not know if it is true at the endpoint $q = p_w^*$ for weights $w \in A_p \setminus A_1$.

We next compare the new optimal exponent p_w^* with the weighted Sobolev exponent q_r defined in (3.9). For a weight $w \in A_r$, we have the relation $q_r \le p_w^*$, since q_r is of the form

$$\frac{1}{p} - \frac{1}{q_r} = \frac{1}{n}\frac{1}{r-\varepsilon},$$

where $\varepsilon > 0$ is given by the left-openness property of the class A_r (see Corollary 1.15). By the definition of ℓ_w we have $\ell_w < r - \varepsilon$, and then we deduce the relation $q_r < p_w^*$ strictly unless $r = 1$,

where $q_1 = p_w^*$. The gap between q_r and p_w^* is given by the precision of the left-openness property: larger ε (i.e. $r - \varepsilon$ closer to ℓ_w) narrows the gap, while smaller ε widens it. Thus, q_r is an approximate optimal exponent given by the left-openness property, whereas p_w^* is the theoretical optimal exponent.

We will prove the following result, which is equivalent to Theorem 3.24.

Proposition 3.25. *We can find a weight $w \in A_p$ such that if the inequality*

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^q w(x)dx\right)^{\frac{1}{q}} \leq C_w \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}} \tag{3.14}$$

holds for every cube Q and for all Lipschitz functions f, where q is defined by

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{n(\ell_w - \gamma)},$$

with γ such that $n(\ell_w - \gamma) > 0$. Then $\gamma \leq 0$.

We will follow the ideas of the proof of Proposition 7.4 from **[PR19]**. We need the following lemma from **[PR19]**.

Lemma 3.26. *Let μ be a finite measure such that* $\operatorname{supp}(\mu) \subset \Omega \subset \mathbb{R}^n$. *Consider a subset $E \subset \Omega$ such that $\mu(E) \geq \lambda\mu(\Omega)$ for some $\lambda \in (0,1)$ and a function f vanishing on E. Then, for any constant $a \in \mathbb{R}$ we have*

$$\|a\|_{L^q(\mu)} \leq \frac{1}{\lambda^{\frac{1}{q}}}\|f - a\|_{L^q(\mu)}.$$

Proof. Since f vanishes on E, we clearly have that $|f(x) - a| = |a|$ for every $x \in E$. Using this fact and the lower bound $\mu(\Omega) \leq \lambda^{-1}\mu(E)$, we can directly estimate the L^q norm as follows,

$$\begin{aligned}
\|a\|_{L^q(\mu)}^q &= |a|^q \mu(\Omega) \\
&\leq |a|^q \frac{\mu(E)}{\lambda} \\
&= \frac{1}{\lambda}\int_E |a|^q \, d\mu \\
&= \frac{1}{\lambda}\int_E |f(x) - a|^q \, d\mu.
\end{aligned}$$

Finally, enlarging the domain of integration from E to Ω, we obtain

$$\|a\|_{L^q(\mu)}^q \leq \frac{1}{\lambda}\int_\Omega |f(x) - a|^q \, d\mu = \frac{1}{\lambda}\|f - a\|_{L^q(\mu)}^q.$$

Taking the $1/q$ power on both sides yields the desired inequality, which completes the proof. □

Proof of Proposition 3.25. Consider $Q = Q(0,1) \subset \mathbb{R}^n$ with $n \geq 2$. Let $1 < p < n$, and consider the weight $w(x) = |x|^{\delta - n}$ with $n < \delta < np$, then $w \in A_p$ (see Example 7.1.7. in **[Gra14]**). Concretely, $\ell_w = \frac{\delta}{n} > 1$, and this implies that $w \in A_r$ for all $r \in (\frac{\delta}{n}, \infty)$.

Before proceeding, let us estimate $w(Q)$. Since $Q = (-1/2, 1/2)^n$, it contains the inscribed ball $B(0, 1/2)$ and is contained in the circumscribed ball $B(0, \sqrt{n}/2)$. Passing to polar coordinates, we can bound the integral from below and above:

$$w(Q) \geq \int_{B(0,1/2)} |x|^{\delta - n} dx = \omega_{n-1}\int_0^{1/2} \rho^{\delta - 1} d\rho = \omega_{n-1}\frac{(1/2)^\delta}{\delta} \simeq_n \frac{1}{\delta},$$

$$w(Q) \leq \int_{B(0,\sqrt{n}/2)} |x|^{\delta-n} dx = \omega_{n-1} \int_0^{\sqrt{n}/2} \rho^{\delta-1} d\rho = \omega_{n-1} \frac{(\sqrt{n}/2)^\delta}{\delta} \simeq_n \frac{1}{\delta}.$$

Hence, we have $w(Q) \simeq_n \frac{1}{\delta}$. Furthermore, by a simple change of variables $x = ry$, we have $w(Q(0,r)) = r^\delta w(Q)$ for any $r > 0$.

Let $0 < \varepsilon < \frac{1}{2}$ be a small number which we will choose later, and let define $E = Q \setminus Q(0, 2\varepsilon)$. We define on Q a piecewise affine Lipschitz function f such that $f(x) = 0$ for all $x \in E$ and $f(x) = 1$ for all $x \in Q(0,\varepsilon)$. The transition region is $Q(0,2\varepsilon) \setminus Q(0,\varepsilon)$, which has a width of $\varepsilon/2$ in the L^∞ distance. Thus, $|\nabla f(x)| \simeq_n \frac{1}{\varepsilon}$ on this region and it is 0 otherwise.

In order to apply the previous lemma, we compute

$$w(E) = w(Q) - w(Q(0,2\varepsilon)) = w(Q) - (2\varepsilon)^\delta w(Q) = \left(1 - (2\varepsilon)^\delta\right) w(Q).$$

Then, applying the lemma and assuming that inequality (3.14) holds, we have

$$\begin{aligned}
\left(\frac{1}{w(Q)} \int_Q |f(x)|^q w(x) dx\right)^{\frac{1}{q}} &\leq \left(\frac{1}{w(Q)} \int_Q |f(x) - f_Q|^q w(x) dx\right)^{\frac{1}{q}} + \left(\frac{1}{w(Q)} \int_Q |f_Q|^q w(x) dx\right)^{\frac{1}{q}} \\
&\leq \left(1 + \frac{1}{\left(1 - (2\varepsilon)^\delta\right)^{\frac{1}{q}}}\right) \left(\frac{1}{w(Q)} \int_Q |f(x) - f_Q|^q w(x) dx\right)^{\frac{1}{q}} \\
&\leq \left(1 + \frac{1}{\left(1 - (2\varepsilon)^\delta\right)^{\frac{1}{q}}}\right) C_w \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x) dx\right)^{\frac{1}{p}}.
\end{aligned}$$

We will study each term separately. For the left-hand side,

$$\begin{aligned}
\left(\frac{1}{w(Q)} \int_Q |f(x)|^q w(x) dx\right)^{\frac{1}{q}} &\geq \left(\frac{1}{w(Q)} \int_{Q(0,\varepsilon)} |f(x)|^q w(x) dx\right)^{\frac{1}{q}} \\
&= \left(\frac{w(Q(0,\varepsilon))}{w(Q)}\right)^{\frac{1}{q}} \\
&= \left(\frac{\varepsilon^\delta w(Q)}{w(Q)}\right)^{\frac{1}{q}} \\
&= \varepsilon^{\frac{\delta}{q}}.
\end{aligned}$$

On the other hand, we bound the gradient term,

$$\begin{aligned}
\ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w(x) dx\right)^{\frac{1}{p}} &\simeq_n \left(\frac{1}{w(Q)} \int_{Q(0,2\varepsilon)\setminus Q(0,\varepsilon)} \frac{1}{\varepsilon^p} w(x) dx\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\varepsilon} \left(\delta \int_{Q(0,2\varepsilon)} |x|^{\delta-n} dx\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\varepsilon} \left(\delta \max_{x \in Q(0,2\varepsilon)} |x|^{\delta-n} |Q(0,2\varepsilon)|\right)^{\frac{1}{p}} \\
&= \frac{1}{\varepsilon} \left(\delta \cdot (\sqrt{n}\varepsilon)^{\delta-n} (2\varepsilon)^n\right)^{\frac{1}{p}} \\
&\simeq_n \delta^{\frac{1}{p}} \varepsilon^{\frac{\delta}{p}-1},
\end{aligned}$$

where we used that the maximum of $|x|^{\delta-n}$ on $Q(0,2\varepsilon) = (-\varepsilon,\varepsilon)^n$ is attained at the vertices, which are at distance $\sqrt{n}\varepsilon$ from the origin, since $\delta - n > 0$.

Hence,

$$\varepsilon^{\frac{\delta}{q}} \lesssim_n C_w \left(1 + \frac{1}{\left(1-(2\varepsilon)^{\delta}\right)^{\frac{1}{q}}}\right) \delta^{\frac{1}{p}} \varepsilon^{\frac{\delta}{p}-1}, \tag{3.15}$$

which implies,

$$1 \lesssim_n C_w \left(1 + \frac{1}{\left(1-(2\varepsilon)^{\delta}\right)^{\frac{1}{q}}}\right) \varepsilon^{\frac{\delta}{p}-\frac{\delta}{q}-1}.$$

Observe that, since $\ell_w = \frac{\delta}{n}$, we have

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{n(\ell_w - \gamma)} = \frac{1}{n\left(\frac{\delta}{n} - \gamma\right)} = \frac{1}{\delta - \gamma n}.$$

Therefore,

$$\begin{aligned} 1 \lesssim_n & C_w \left(1 + \frac{1}{\left(1-(2\varepsilon)^{\delta}\right)^{\frac{1}{q}}}\right) \varepsilon^{\frac{\delta}{p}-\frac{\delta}{q}-1} \\ = & C_w \left(1 + \frac{1}{\left(1-(2\varepsilon)^{\delta}\right)^{\frac{1}{q}}}\right) \varepsilon^{\frac{\delta}{\delta-\gamma n}-1} \\ = & C_w \left(1 + \frac{1}{\left(1-(2\varepsilon)^{\delta}\right)^{\frac{1}{q}}}\right) \varepsilon^{\frac{\gamma n}{\delta-\gamma n}}. \end{aligned}$$

Since $\delta - \gamma n > 0$ by hypothesis, if $\gamma > 0$, we can choose $\varepsilon > 0$ small enough such that the previous inequality fails. This forces $\gamma \le 0$, and the proof is complete. □

The following result is a generalization of Proposition 7.4 in **[PR19]**.

Proposition 3.27. *Let $1 < p < n$ and $1 \le r \le p \le q < \infty$. Suppose that there exist constants $C > 0$ and $\beta \ge 0$ such that*

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^q w(x)\,dx\right)^{\frac{1}{q}} \le C[w]_{A_r}^{\beta}\,\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)\,dx\right)^{\frac{1}{p}}$$

holds for every $w \in A_r$, every cube Q, and every admissible function f. Then $\beta \ge \frac{1}{p}$.

As a consequence of this result, we obtain the sharpness of the exponent $\frac{1}{p}$ on the A_r constant in (3.10).

Proof. Let $Q = Q(0,1)$ and consider the weight $w(x) = |x|^{\delta-n}$ with $0 < \delta < \min\{1, nr\}$. Then $w \in A_r$ and $[w]_{A_r} \simeq_n \frac{n^r}{\delta}\left(\frac{r-1}{nr-\delta}\right)^{r-1}$. Moreover, by homogeneity, $w(tQ) = t^{\delta} w(Q)$ for every $t > 0$. Fix $\varepsilon \in (0, \frac{1}{4})$ such that $(2\varepsilon)^n \le \frac{1}{4}$, and let f be a Lipschitz function such that $0 \le f \le 1$, $f = 1$ on εQ, $f = 0$ outside $2\varepsilon Q$, and $|\nabla f| \lesssim_n \frac{1}{\varepsilon}$. Since $|Q| = 1$, we have $0 \le f_Q \le (2\varepsilon)^n \le \frac{1}{4}$. Therefore,

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^q w(x)\,dx\right)^{\frac{1}{q}} \ge (1 - f_Q)\left(\frac{w(\varepsilon Q)}{w(Q)}\right)^{\frac{1}{q}} \ge \frac{3}{4}\varepsilon^{\delta/q}.$$

On the other hand, since $2^\delta - 1 \le \delta$ for $0 < \delta < 1$, we have

$$\begin{aligned}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)\,dx\right)^{\frac{1}{p}} &\lesssim_n \frac{1}{\varepsilon}\left(\frac{w(2\varepsilon Q) - w(\varepsilon Q)}{w(Q)}\right)^{\frac{1}{p}} \\ &= \varepsilon^{\frac{\delta}{p}-1}(2^\delta - 1)^{\frac{1}{p}} \\ &\le \delta^{\frac{1}{p}}\varepsilon^{\frac{\delta}{p}-1}.\end{aligned}$$

Applying the assumed inequality and dividing by $\varepsilon^{\delta/q}$, we obtain

$$\begin{aligned}1 &\lesssim_n C\left(\frac{n^r}{\delta}\left(\frac{r-1}{nr-\delta}\right)^{r-1}\right)^\beta \delta^{\frac{1}{p}}\varepsilon^{\frac{\delta}{p}-\frac{\delta}{q}-1} \\ &= Cn^{r\beta}\left(\frac{r-1}{nr-\delta}\right)^{(r-1)\beta}\frac{\varepsilon^{\frac{\delta}{p}-\frac{\delta}{q}-1}}{\delta^{\beta-\frac{1}{p}}}.\end{aligned}$$

For the fixed value of ε, if we take the limit as $\delta \to 0^+$, the term $\delta^{\beta-\frac{1}{p}}$ in the denominator forces the condition $\beta \ge \frac{1}{p}$. This completes the proof. □

3.6 Asymptotically sharp embedding of A_∞ into A_p for flat weights

The purpose of this section is to provide new quantitative estimates for the embedding of A_∞ weights into A_p classes. A classical result is that the A_p classes are increasing in p and *open*, in the sense that if $w \in A_p$ with $p > 1$ then there exists an $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$, with an explicit control of the $A_{p-\varepsilon}$ constant $[w]_{A_{p-\varepsilon}}$ in terms of $[w]_{A_p}$ (see Corollaries 1.15 and 1.18). The relevant question here is to determine, given the weight $w \in A_\infty$ with a certain constant $[w]_{A_\infty}$, what is the best possible $p \ge 1$ such that $w \in A_p$ (this is ℓ_w in Definition 3.5). Moreover, it is interesting to find a quantitative connection between this optimal $p \ge 1$ and the Fujii–Wilson constant $[w]_{A_\infty}$.

Work in this direction can be found in **[HP16]**, where, as a consequence of their study of quantitative weighted Solyanik estimates, the authors prove that if w is an A_∞ weight in $\mathbb{R}^n$, then $w \in A_p$ for $p > e^{c_n[w]_{A_\infty}}$ with $[w]_{A_p} \le e^{e^{c_n[w]_{A_\infty}}}$, where $c_n > 1$ is a dimensional constant. This seems to be the best available result in the literature on this question, but a simple observation is that both these estimates fall short of what they should be in the particular case of weights with $[w]_{A_\infty} \to 1$. Notice that since $[w]_{A_\infty} = 1$ is equivalent to w being constant, it makes sense to refer to this asymptotic analysis as the case of *flat weights*. The main contribution of this section is to provide a sharp asymptotic estimate for the $A_\infty \hookrightarrow A_p$ embedding for flat weights. That is, denoting by $p([w]_{A_\infty})$ the index such that $w \in A_{p([w]_{A_\infty})}$ we show that $p([w]_{A_\infty}) \to 1$ when $[w]_{A_\infty} \to 1$, with explicit quantitative estimates for the A_p constant. The remarkable aspect is that we indeed need to restrict ourselves to weights with small A_∞ constants, a condition that seems to be common in these types of results. We do not know whether similar estimates can be proved for *all* A_∞ weights.

We now state the main result of this section.

Theorem 3.28. *There exist dimensional constants $c_n, \tau_n, \tau_n' > 0$ such that if $w \in A_\infty$ with $[w]_{A_\infty} \le 1 + c_n$ then*

$$w \in A_p \qquad \text{with} \qquad p = 1 + \tau_n([w]_{A_\infty} - 1),$$

and moreover,

$$[w]_{A_p} \leq \exp(\tau_n'([w]_{A_\infty} - 1)).$$

Remark 3.29. We observe that under the smallness condition $[w]_{A_\infty} \leq 1 + c_n$ of the preceding theorem, the exponential upper bound can be controlled linearly. As a result, the quantities p and $[w]_{A_p}$ exhibit the same asymptotic behavior, yielding the relation

$$p \simeq_n [w]_{A_p} \simeq_n 1 + \tau_n([w]_{A_\infty} - 1).$$

3.6.1 Logarithm of an A_∞ weight

A key intermediate step towards the proof of Theorem 3.28 is an estimate on the BMO seminorm of $\log w$, where w is an A_∞ weight. The fact that $\log w$ belongs to BMO when $w \in A_\infty$ is not new; even some quantitative estimates are known. A nice result in this area was provided by Korey **[Kor98]** related to the $[w]'_{A_\infty}$ constant. Among other results, Korey proved that

$$\| \log w\|_{\mathrm{BMO}} \leq \log \left(2[w]'_{A_\infty}\right).$$

There is also an asymptotic estimate of the form

$$\| \log w\|_{\mathrm{BMO}} \leq C\sqrt{\log \left([w]'_{A_\infty}\right)} \qquad \text{as} \qquad [w]_{A_\infty} \to 1.$$

Results like these were later used by Pattakos and Volberg in **[PV12]** to study the interesting problem of the continuity of norm estimates in weighted inequalities for Calderón–Zygmund operators.

In that direction, we provide here a sharper estimate for the BMO seminorm in terms of a mixed combination of the Hruščev and Fujii–Wilson A_∞ constants. This estimate should be compared to the result in **[HP13, Proposition 1.21]**.

Theorem 3.30. *If $w \in A_\infty$, then $\log w \in$ BMO with*

$$\|\log w\|_{\mathrm{BMO}} \leq c_n \log(2[w]'_{A_\infty})([w]_{A_\infty} - 1),$$

for some dimensional constant $c_n > 0$.

For this purpose, we introduce the following logarithmic A_∞ constant to be used throughout the section.

$$[w]^{\log}_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q \left(1 + \log^+ \left(\frac{w(x)}{w_Q}\right)\right) w(x)dx,$$

where the supremum is taken over all cubes Q.

Remark 3.31. It is shown in **[HP13]** that $[w]^{\log}_{A_\infty} \leq 2^{n+1}[w]_{A_\infty}$. In addition, by **[Ste69]** we have the reverse inequality,

$$\int_Q \mathcal{M}(w\chi_Q)(x)dx \leq 3^n \int_Q \left(1 + \log^+ \left(\frac{w(x)}{w_Q}\right)\right) w(x)\, dx,$$

for each cube Q. In conclusion, we have $[w]^{\log}_{A_\infty} \simeq_n [w]_{A_\infty}$.

As with the other A_∞ constants, this logarithmic counterpart attains its minimum value 1 precisely for constant weights.

Proposition 3.32. *Let w be a weight. Then $[w]_{A_\infty}^{\log} = 1$ if and only if w is constant almost everywhere.*

Proof. If w is constant almost everywhere, then $w(x) = w_Q$ for almost every $x \in Q$, and hence $\log^+(w(x)/w_Q) = 0$ almost everywhere on Q. Therefore

$$\frac{1}{w(Q)} \int_Q \left(1 + \log^+\left(\frac{w(x)}{w_Q}\right)\right) w(x)\,dx = 1,$$

for every cube Q, so $[w]_{A_\infty}^{\log} = 1$. Conversely, assume that $[w]_{A_\infty}^{\log} = 1$. Hence we must have, for every cube Q,

$$\int_Q \log^+\left(\frac{w(x)}{w_Q}\right) w(x)\,dx = 0.$$

Thus $\log^+(w(x)/w_Q) = 0$ for almost every $x \in Q$, and therefore $w(x) \le w_Q$ for almost every $x \in Q$. It follows that $w_Q - w(x) \ge 0$ almost everywhere on Q, while

$$\int_Q (w_Q - w(x))\,dx = |Q|\, w_Q - \int_Q w(x)\,dx = 0.$$

Hence $w(x) = w_Q$ for almost every $x \in Q$. Since this holds for every cube Q, then w is constant almost everywhere. □

As a first step, we show that $\log w$ belongs to the weighted BMO space, BMO_w. This space, which is usually denoted as $\mathrm{BMO}_{w,w}$ in the literature (see, for instance, **[OPRR20]**), is equipped with the seminorm

$$\|f\|_{\mathrm{BMO}_w} := \sup_Q \frac{1}{w(Q)} \int_Q |f(x) - f_{Q,w}| w(x)\,dx,$$

where the supremum is taken over all cubes Q, and $f_{Q,w} = \frac{1}{w(Q)} \int_Q f(x)w(x)\,dx$ is the weighted average of f on the cube Q.

Proposition 3.33. *If $w \in A_\infty$, then $\log w \in \mathrm{BMO}_w$ and*

$$\|\log w\|_{\mathrm{BMO}_w} \le 8([w]_{A_\infty}^{\log} - 1). \tag{3.16}$$

Proof. Let $w \in A_\infty$ and let Q be a cube. Since $\log w - (\log w)_{Q,w}$ has zero integral on the cube Q with respect to the weighted measure $w\,dx$, we have

$$\begin{aligned}
\frac{1}{w(Q)} \int_Q |\log w - (\log w)_{Q,w}| w &= \frac{2}{w(Q)} \int_Q \left(\log w - (\log w)_{Q,w}\right)^+ w \\
&\le \frac{2}{w(Q)} \int_Q \left(\log w - \log w_Q\right)^+ w \\
&\quad + 2\left(\log w_Q - (\log w)_{Q,w}\right)^+ \\
&= I_1 + I_2,
\end{aligned}$$

where $a^+ = \max(a, 0)$. We can easily estimate the first term

$$\begin{aligned} I_1 =& \frac{2}{w(Q)} \int_Q \log^+ \left(\frac{w(x)}{w_Q} \right) w(x)\, dx \\ =& 2 \left(\frac{1}{w(Q)} \int_Q \left(1 + \log^+ \left(\frac{w(x)}{w_Q} \right) \right) w(x)\, dx - 1 \right) \\ \leq& 2([w]_{A_\infty}^{\log} - 1). \end{aligned}$$

On the other hand,

$$\begin{aligned} (\log w)_{Q,w} - \log w_Q =& \frac{1}{w(Q)} \int_Q (\log w(x) - \log w_Q) w(x)\, dx \\ =& \frac{1}{w(Q)} \int_Q (\log w(x) - \log w_Q)^+ w(x)\, dx \\ & - \frac{1}{w(Q)} \int_Q (\log w_Q - \log w(x))^+ w(x)\, dx \\ =& P - N. \end{aligned}$$

We estimate P and N separately. The bound for P is analogous to that for I_1,

$$\begin{aligned} P =& \frac{1}{w(Q)} \int_Q \log^+ \left(\frac{w(x)}{w_Q} \right) w(x)\, dx \\ =& \frac{1}{w(Q)} \int_Q \left(1 + \log^+ \left(\frac{w(x)}{w_Q} \right) \right) w(x)\, dx - 1 \\ \leq& [w]_{A_\infty}^{\log} - 1. \end{aligned}$$

To estimate N, we use the elementary inequality $\log t \leq t - 1$,

$$\begin{aligned} N =& \frac{1}{w(Q)} \int_Q \left(\log \left(\frac{w_Q}{w(x)} \right) \right)^+ w(x)\, dx \\ \leq& \frac{1}{w(Q)} \int_{\{w \leq w_Q\}} \left(\frac{w_Q}{w(x)} - 1 \right) w(x)\, dx \\ =& \frac{1}{w(Q)} \int_Q \left(w_Q - w(x) \right)^+ dx \\ \leq& \frac{1}{w(Q)} \int_Q |w(x) - w_Q|\, dx \\ =& \frac{2}{w(Q)} \int_Q (w(x) - w_Q)^+ dx \\ =& \frac{2}{w(Q)} \int_{\{w \geq w_Q\}} w_Q \left(\frac{w(x)}{w_Q} - 1 \right) dx \\ \leq& \frac{2}{w(Q)} \int_{\{w \geq w_Q\}} w_Q \frac{w(x)}{w_Q} \log \left(\frac{w(x)}{w_Q} \right) dx \\ =& \frac{2}{w(Q)} \int_Q \log^+ \left(\frac{w(x)}{w_Q} \right) w(x)\, dx \\ \leq& 2([w]_{A_\infty}^{\log} - 1), \end{aligned}$$

where in the third inequality we use $t - 1 \le t \log t$ for $t \ge 1$. Therefore,

$$\begin{aligned} I_2 =& 2\left(\log w_Q - (\log w)_{Q,w}\right)^+ \\ \le& 2|\log w_Q - (\log w)_{Q,w}| \\ \le& 2|(\log w)_{Q,w} - \log w_Q| \\ =& 2|P - N| \\ \le& 2P + 2N \\ \le& 2([w]_{A_\infty}^{\log} - 1) + 4([w]_{A_\infty}^{\log} - 1) \\ =& 6([w]_{A_\infty}^{\log} - 1). \end{aligned}$$

Combining the previous estimates, we have

$$\frac{1}{w(Q)} \int_Q |\log w - (\log w)_{Q,w}| w \le 8([w]_{A_\infty}^{\log} - 1),$$

for every cube Q. Taking the supremum over all cubes Q completes the proof. □

Remark 3.34. A closer inspection of the argument shows that we have, in fact, proved a stronger statement. For each cube Q, we have

$$\frac{1}{w(Q)} \int_Q |\log w(x) - (\log w)_{Q,w}| w(x)\, dx \le \frac{8}{w(Q)} \int_Q \log^+\left(\frac{w(x)}{w_Q}\right) w(x)\, dx.$$

The estimate obtained above has the correct asymptotic behavior for flat weights (see Proposition 3.32), in the sense that as $[w]_{A_\infty}^{\log} \to 1$, both sides of the inequality go to zero. However, if we simply used Lemma 6.2 from **[HP13]**, which states that $[w]_{A_\infty}^{\log} \le 2^{n+1}[w]_{A_\infty}$, we would lose this property. Indeed, such a bound would yield $\|\log w\|_{\mathrm{BMO}_w} \le c_n[w]_{A_\infty} - 1$, which clearly does not vanish as $[w]_{A_\infty} \to 1$. Therefore, we need to refine this result to obtain an inequality that preserves the correct asymptotic behavior for flat weights.

Lemma 3.35. *For any cube Q and any measurable function w, we have*

$$\frac{1}{w(Q)} \int_Q \log^+\left(\frac{w(x)}{w_Q}\right) w(x) dx \le 2^n \left(\frac{1}{w(Q)} \int_Q \mathcal{M}(w\chi_Q)(x) dx - 1\right).$$

Hence, if $w \in A_\infty$,

$$[w]_{A_\infty}^{\log} - 1 \le 2^n([w]_{A_\infty} - 1).$$

Proof. The estimate follows from the reverse weak type (1,1) inequality: if w is nonnegative and $t > w_Q$,

$$\frac{1}{t} \int_{\{x \in Q:\ w(x) > t\}} w\, dx \le 2^n \left|\{x \in Q :\ \mathcal{M}(w\chi_Q)(x) > t\}\right|.$$

By the layer-cake formula applied to $s \log^+(\frac{s}{w_Q})$, we have

$$\begin{aligned}\frac{1}{w(Q)}\int_Q \log^+\left(\frac{w}{w_Q}\right) w &= \frac{1}{w(Q)}\int_{w_Q}^{\infty}\frac{1}{t}\left(\int_{\{x\in Q:\, w(x)>t\}} w\,dx\right)dt\\ &\le \frac{2^n}{w(Q)}\int_{w_Q}^{\infty}\left|\{x\in Q:\ \mathcal{M}(w\chi_Q)(x)>t\}\right|\,dt\\ &= \frac{2^n}{w(Q)}\int_{0}^{\infty}\left|\{x\in Q:\ \mathcal{M}(w\chi_Q)(x)>t\}\right|\,dt\\ &\quad - \frac{2^n}{w(Q)}\int_{0}^{w_Q}\left|\{x\in Q:\ \mathcal{M}(w\chi_Q)(x)>t\}\right|\,dt\\ &= \frac{2^n}{w(Q)}\int_Q \mathcal{M}(w\chi_Q)(x)\,dx\\ &\quad - \frac{2^n}{w(Q)}\int_{0}^{w_Q}\left|\{x\in Q:\ \mathcal{M}(w\chi_Q)(x)>t\}\right|\,dt.\end{aligned}$$

We note that since $w_Q \le \mathcal{M}(w\chi_Q)(x)$ for each $x \in Q$, we have

$$\begin{aligned}&\frac{2^n}{w(Q)}\int_{0}^{w_Q}\left|\{x\in Q:\ \mathcal{M}(w\chi_Q)(x)>t\}\right|\,dt\\ &\quad\ge \frac{2^n}{w(Q)}\int_{0}^{w_Q}\left|\{x\in Q:\ \mathcal{M}(w\chi_Q)(x)>w_Q\}\right|\,dt\\ &\quad= \frac{2^n}{w(Q)}|Q|\int_{0}^{w_Q}dt\\ &\quad= 2^n\frac{|Q|}{w(Q)}w_Q\\ &\quad= 2^n.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{w(Q)}\int_Q\left(1+\log^+\left(\frac{w}{w_Q}\right)\right)w\,dx - 1 &= \frac{1}{w(Q)}\int_Q \log^+\left(\frac{w}{w_Q}\right)w\,dx\\ &\le 2^n\left(\frac{1}{w(Q)}\int_Q \mathcal{M}(w\chi_Q)\,dx - 1\right),\end{aligned}$$

and we conclude the proof by taking the supremum over Q. □

Combining the two previous results, we obtain the bound of the BMO_w seminorm in terms of the Fujii–Wilson A_∞ constant.

Corollary 3.36. *If $w \in A_\infty$, then $\log w \in \mathrm{BMO}_w$ and*

$$\|\log w\|_{\mathrm{BMO}_w} \le 2^{n+3}([w]_{A_\infty} - 1). \tag{3.17}$$

In addition,

$$\frac{1}{w(Q)}\int_Q |\log w(x) - (\log w)_{Q,w}|w(x)\,dx \le 2^{n+3}\left(\frac{1}{w(Q)}\int_Q \mathcal{M}(w\chi_Q)(x)\,dx - 1\right),$$

for each cube Q.

The following result is due to Tsutsui **[Tsu14]**. However, for the convenience of the reader and the sake of completeness, we provide a detailed proof here.

Theorem 3.37. *There exists a dimensional constant $c_n > 0$ such that for any $w \in A_\infty$,*

$$\|f\|_{\mathrm{BMO}} \leq c_n \log(2[w]'_{A_\infty}) \|f\|_{\mathrm{BMO}_w},$$

where $[w]'_{A_\infty}$ is the Hruščev A_∞ constant defined in (1.10).

Proof. Fix a cube Q. By standard BMO arguments, we have

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \leq 2 \inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(x)-c|\,dx \leq \frac{2}{|Q|}\int_Q |f(x) - f_{Q,w}|\,dx.$$

Next, by the characterization of $[w]'_{A_\infty}$ given in **[Gra14, Proposition 7.3.2 (5)]**, for every non-negative measurable function g we have

$$\exp\left(\frac{1}{|Q|}\int_Q g(x)\,dx\right) \leq [w]'_{A_\infty} \frac{1}{w(Q)}\int_Q \exp(g(x))w(x)\,dx.$$

On the other hand, by the definition of the Luxemburg norm, any measurable function h satisfies

$$\frac{1}{w(Q)}\int_Q \exp\left(\frac{|h(x)|}{\|h\|_{\exp L(Q,w)}}\right) w(x)\,dx \leq 2.$$

We now apply these two facts with

$$g(x) = \frac{|f(x) - f_{Q,w}|}{\|f - f_{Q,w}\|_{\exp L(Q,w)}},$$

and $h(x) = f(x) - f_{Q,w}$, hence we obtain

$$\exp\left(\frac{1}{|Q|}\int_Q \frac{|f(x) - f_{Q,w}|}{\|f - f_{Q,w}\|_{\exp L(Q,w)}}\,dx\right) \leq 2[w]'_{A_\infty}.$$

Taking logarithms gives

$$\frac{1}{|Q|}\int_Q |f(x) - f_{Q,w}|\,dx \leq \log(2[w]'_{A_\infty})\,\|f - f_{Q,w}\|_{\exp L(Q,w)}.$$

Finally, we apply the John–Nirenberg theorem for non-doubling measures due to Mateu, Mattila, Nicolau, and Orobitg **[MMNO00, Theorem 1]**[1]. It follows that there exists a dimensional constant $c_n > 0$ such that

$$\|f - f_{Q,w}\|_{\exp L(Q,w)} \leq c_n \|f\|_{\mathrm{BMO}_w}.$$

Combining the previous estimates, we conclude that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \leq 2c_n \log(2[w]'_{A_\infty})\,\|f\|_{\mathrm{BMO}_w}.$$

Taking the supremum over all cubes Q completes the proof. □

As a consequence of the results above, we obtain the result on the BMO norm of $\log w$ presented in Theorem 3.30.

Proof of Theorem 3.30. This is just putting together (3.17) from Corollary 3.36 and Tsutsui's result in Theorem 3.37. □

[1] As explicitly stated at the end of the proof of Theorem 1 in **[MMNO00]**, the constants in the John–Nirenberg inequality depend only on the dimension and, in particular, are independent of the measure μ.

3.6.2 Proof of Theorem 3.28 and its consequences

Using the results established in the previous section, we now proceed with the proof of Theorem 3.28. Our approach follows the technique of **[Pol95]** (see also **[Kor98; Mit05]**).

Proof of Theorem 3.28. The goal is to obtain control on the A_p condition for $w \in A_\infty$ by assuming some quantitative control on $[w]_{A_\infty}$. We begin by considering $r \geq 1$ to be chosen later and connect the A_p condition with previous estimates on the BMO norm of $\log w$.

Let Q be a cube. By Jensen's inequality and the elementary inequality $\exp(t) \leq \exp(|t|)$ we obtain

$$\begin{aligned}
\frac{1}{|Q|}\int_Q w\left(\frac{1}{|Q|}\int_Q w^{-r}\right)^{\frac{1}{r}} &\leq \left(\frac{1}{|Q|}\int_Q w^{r}\right)^{\frac{1}{r}}\left(\frac{1}{|Q|}\int_Q w^{-r}\right)^{\frac{1}{r}}\\
&= \left(\frac{1}{|Q|}\int_Q e^{r\log w}\right)^{\frac{1}{r}}\left(\frac{1}{|Q|}\int_Q e^{-r\log w}\right)^{\frac{1}{r}}\\
&= \left(\frac{1}{|Q|}\int_Q e^{r(\log w-(\log w)_Q)}\right)^{\frac{1}{r}}\left(\frac{1}{|Q|}\int_Q e^{-r(\log w-(\log w)_Q)}\right)^{\frac{1}{r}}\\
&\leq \left(\frac{1}{|Q|}\int_Q e^{|r||\log w-(\log w)_Q|}\right)^{\frac{1}{r}}\left(\frac{1}{|Q|}\int_Q e^{|-r||\log w-(\log w)_Q|}\right)^{\frac{1}{r}}\\
&= \left(\frac{1}{|Q|}\int_Q e^{r|\log w-(\log w)_Q|}\right)^{\frac{2}{r}}.
\end{aligned}$$

This last quantity is exactly what we know how to control by using the BMO estimates for $\log w$. More precisely, by Theorem 3.30 we have $\log w \in \mathrm{BMO}$, and by John–Nirenberg inequality we can obtain that

$$|\{x \in Q : \ |\log w(x) - (\log w)_Q| > t\}| \leq 2\exp\left(-\frac{\alpha_n t}{\|\log w\|_{\mathrm{BMO}}}\right)|Q|.$$

Hence,

$$\frac{1}{|Q|}\int_Q \exp\left(r|\log w(x) - (\log w)_Q|\right)\, dx \leq 3, \tag{3.18}$$

where

$$r = \frac{\alpha_n}{2\|\log w\|_{\mathrm{BMO}}}.$$

We use estimate in (3.18) to control the last integral in our previous estimate to obtain

$$\frac{1}{|Q|}\int_Q w\left(\frac{1}{|Q|}\int_Q w^{-r}\right)^{\frac{1}{r}} \leq 3^{\frac{2}{r}}. \tag{3.19}$$

It remains to verify two points. On one hand, we need to show that r above can be chosen such that $r \geq 1$. This will be a consequence of the estimates on $\|\log w\|_{\mathrm{BMO}}$. On the other hand, we need to exploit this result to obtain an explicit expression for $p([w]_{A_\infty})$, the best possible index for which $w \in A_p$.

For the first one, we simply need to restrict to weights with small constant, $[w]_{A_\infty} = 1 + \delta$ with δ small enough such that $\|\log w\|_{\mathrm{BMO}} \leq \frac{\alpha_n}{2}$. To that end, recall that we have the estimate from Theorem 3.30, namely

$$\|\log w\|_{\mathrm{BMO}} \leq c_n \log(2[w]'_{A_\infty})([w]_{A_\infty} - 1).$$

Since our standing assumption is that $[w]_{A_\infty} = 1 + \delta$ with a small δ, we may assume that $[w]_{A_\infty}$ is bounded by, let's say, 2. From that, we want to obtain that $[w]'_{A_\infty}$ is controlled as well. This can be found in **[BR14, Theorem 1.3]**, where the inequality

$$[w]'_{A_\infty} \leq C \frac{e^{e^{[w]_{A_\infty}+1}}}{e^{[w]_{A_\infty}}}$$

is proved in dimension 1. For the higher dimensional case in $\mathbb{R}^n$, $n \geq 2$, the result is from **[HP16, Corollary 5.6]** where the inequality

$$[w]'_{A_\infty} \leq e^{e^{c_n [w]_{A_\infty}}}$$

has been proved. Hence, under the assumption $[w]_{A_\infty} = 1 + \delta$ with $\delta \leq 1$, we have

$$\|\log w\|_{\mathrm{BMO}} \leq C_n([w]_{A_\infty} - 1).$$

We now choose $\delta \leq \frac{\alpha_n}{2C_n}$ to get the desired estimate $\| \log w\|_{\mathrm{BMO}} \leq \frac{\alpha_n}{2}$.

To finish with the proof we now consider $p - 1 = \frac{1}{r}$, i.e. $p = 1 + \frac{1}{r}$. This yields

$$\frac{1}{|Q|} \int_Q w \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} \leq 3^{2(p-1)}.$$

We have proved $[w]_{A_p} \leq 3^{2(p-1)}$ with $p = 1 + \frac{2C_n}{\alpha_n}([w]_{A_\infty} - 1)$. Using this expression for p, we have

$$[w]_{A_p} \leq 3^{2(p-1)} = 3^{\frac{4C_n}{\alpha_n}([w]_{A_\infty}-1)},$$

which is the desired estimate with $\tau_n = \frac{2C_n}{\alpha_n}$ and $\tau'_n = 2 \log 3 \ \tau_n$. This concludes the proof. □

The only result of this type, a quantitative embedding of A_∞ into some A_p class, that we are aware of in the literature is the following theorem from **[HP16]**.

Theorem 3.38. *If $w \in A_\infty$ then $w \in A_p$ with $p = e^{c_n [w]_{A_\infty}}$ and $[w]_{A_p} \leq e^{e^{c_n [w]_{A_\infty}}}$ for some dimensional constant $c_n > 1$.*

Remark 3.39. We note that the corresponding result in **[HP16]** is originally formulated with the strict inequality $p > e^{c_n [w]_{A_\infty}}$. However, one can readily obtain the statement with equality, $p = e^{c_n [w]_{A_\infty}}$, by replacing c_n with a slightly larger dimensional constant.

Our result improves this estimate in the case $[w]_{A_\infty} \leq 1 + c_n$, yielding the correct asymptotic behavior for flat weights. That is, when $[w]_{A_\infty} \to 1$ we have $p \to 1$ and $[w]_{A_p} \to 1$. Putting together our result with the embedding of **[HP16]**, we obtain the following quantitative general embedding.

Corollary 3.40. *Let $w \in A_\infty$. Then $w \in A_p$ with*

$$p = \begin{cases} 1 + \tau_n([w]_{A_\infty} - 1) & \text{if } [w]_{A_\infty} \leq 1 + c_n, \\ e^{C_n [w]_{A_\infty}} & \text{if } [w]_{A_\infty} > 1 + c_n. \end{cases}$$

for some dimensional constant $c_n > 0$. In addition,

$$[w]_{A_p} \leq \begin{cases} e^{\tau'_n([w]_{A_\infty}-1)} & \text{if } [w]_{A_\infty} \leq 1 + c_n, \\ e^{e^{C_n [w]_{A_\infty}}} & \text{if } [w]_{A_\infty} > 1 + c_n. \end{cases}$$

Remark 3.41. The restriction $[w]_{A_\infty} \le 1 + c_n$ is natural in our approach. It is required to guarantee that the integrability exponent r obtained from the John–Nirenberg inequality is large enough (specifically, $r \ge 1$), which is a necessary condition for our estimates. However, one may conjecture that even for large A_∞ constants, the embedding could still hold with p of the order $C_n[w]_{A_\infty}$.

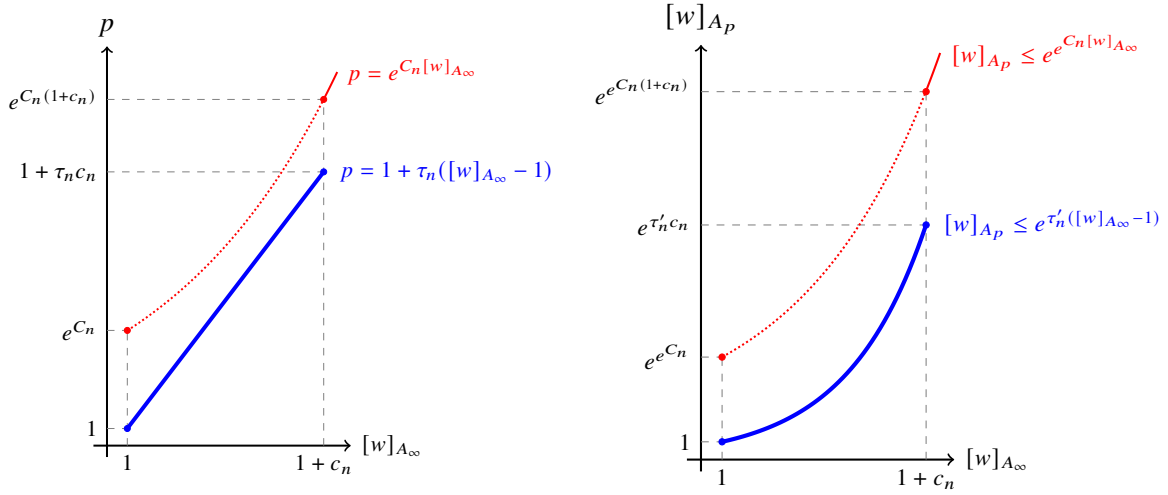


Figure 3.1: Asymptotic behavior of the $A_\infty \hookrightarrow A_p$ embedding. **Left:** The function $p([w]_{A_\infty})$. **Right:** The A_p constant bound. In both cases, the blue line represents the sharp behavior for flat weights obtained in Theorem 3.28, and the red line is the exponential bound from **[HP16]** stated in Theorem 3.38.

As a further consequence of our asymptotic embedding, we obtain a uniform bound of the classical BMO norm by the weighted BMO_w norm for flat weights (see Theorem 3.37).

Corollary 3.42. *There exist dimensional constants $c_n, C_n > 0$ such that if $w \in A_\infty$ with $[w]_{A_\infty} \le 1 + c_n$, then*

$$\|f\|_{\mathrm{BMO}} \le C_n \|f\|_{\mathrm{BMO}_w}.$$

Proof. Let c_n and τ_n be the dimensional constants given by Theorem 3.28. If $[w]_{A_\infty} \le 1 + c_n$, we know that $w \in A_p$ with $p = 1 + \tau_n([w]_{A_\infty} - 1)$, and its A_p constant is uniformly bounded by a dimensional constant, say $[w]_{A_p} \le \tilde{C}_n$. Furthermore, observe that since $[w]_{A_\infty} \le 1 + c_n$, the doubling constant is also bounded by a dimensional constant.

Fix a cube Q. Using Hölder's inequality together with the A_p condition, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx &\le [w]_{A_p}^{\frac{1}{p}} \left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^p w(x)dx\right)^{\frac{1}{p}} \\
&\le [w]_{A_p}^{\frac{1}{p}} C_n D_w p \|f\|_{\mathrm{BMO}_w} \\
&\le C_n \|f\|_{\mathrm{BMO}_w}.
\end{aligned}$$

The second inequality follows from **[MRR22, Theorem 1.1]**, which bounds the $L^p(w)$ average of the oscillation by the BMO_w norm. Finally, the last inequality holds because the parameters p, $[w]_{A_p}$, and D_w are all bounded by dimensional constants. □

3.7 Poincaré–Sobolev inequalities for flat A_∞ weights

In this section, we apply our asymptotic embedding result (Theorem 3.28) to prove weighted Poincaré–Sobolev inequalities for A_∞ weights with a small A_∞ constant. Recall from Section 3.5 that the critical index $\ell_w = \inf\{r \geq 1 : w \in A_r\}$ dictates the optimal weighted Sobolev exponent p_w^* through the relation

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n}\frac{1}{\ell_w}.$$

For a general $w \in A_\infty$, the index ℓ_w can be arbitrarily large, which is why the Poincaré inequality fails for the full A_∞ class. However, our embedding theorem provides an explicit, sharp upper bound for this critical index when the weight is flat: there exists a dimensional constant $c_n > 0$ such that if $[w]_{A_\infty} \leq 1 + c_n$, then $\ell_w \leq 1 + \tau_n([w]_{A_\infty} - 1)$.

We are now ready to state the main result of this section.

Theorem 3.43. *Let $n \geq 2$ and let $1 < p < n$. Let $w \in A_\infty$ and consider the exponent $p_{A_\infty}^*$ defined by the relation*

$$\frac{1}{p} - \frac{1}{p_{A_\infty}^*} = \frac{1}{n}\frac{1}{1 + \tau_n([w]_{A_\infty} - 1)}.$$

There exists $c_{n,p} > 0$ such that if $[w]_{A_\infty} \leq 1 + c_{n,p}$, then we have

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^{p_{A_\infty}^*} w\right)^{\frac{1}{p_{A_\infty}^*}} \leq c_n p^* (1 + \tau_n([w]_{A_\infty} - 1))^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}},$$

for every cube Q. Moreover, we have

$$\left\|f - f_Q\right\|_{L^{p_{A_\infty}^*,p}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq c_n p^* (1 + \tau_n([w]_{A_\infty} - 1))^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}},$$

for every cube Q.

Although the previous result is restricted to weights with a small A_∞ constant, it has the remarkable consequence of asymptotically recovering the classical Sobolev exponent when $[w]_{A_\infty} \to 1$. That is, when $[w]_{A_\infty} \to 1$, we have $p_{A_\infty}^* \to p^* = \frac{np}{n-p}$.

Proof of Theorem 3.43. Let $w \in A_\infty$ with $[w]_{A_\infty} \leq 1 + c_n$, where c_n is given by Theorem 3.28. Then, $w \in A_r$ with $r = 1 + \tau_n([w]_{A_\infty} - 1)$ and $[w]_{A_r} \lesssim_n 1 + \tau_n([w]_{A_\infty} - 1)$. We choose $c_{n,p}$ as the minimum between c_n and the constant depending on n and p ensuring that $r \leq p$, that is, $c_{n,p} \leq \min\{c_n, (p-1)\tau_n^{-1}\}$. Hence $w \in A_p$, and we can apply Proposition 3.10 with $\alpha = 1$ and this value of r, obtaining $q_r = p_{A_\infty}^*$. For each cube Q, using the subrepresentation formula stated in Theorem 1.28, we have

$$\begin{aligned}
\left\|f - f_Q\right\|_{L^{p_{A_\infty}^*,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} &\leq c_n \left\|I_1(|\nabla f|\chi_Q)\right\|_{L^{q_r,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \\
&\leq C_n p^* [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}} \\
&\leq C_n p^* (1 + \tau_n([w]_{A_\infty} - 1))^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}.
\end{aligned}$$

Finally, we conclude the proof by applying the truncation argument stated in Theorem 1.35. □

Remark 3.44. Observe that, since the theorem assumes the smallness condition $[w]_{A_\infty} \leq 1 + c_{n,p}$, the term $(1 + \tau_n([w]_{A_\infty} - 1))^{\frac{1}{p}}$ is uniformly bounded and could be absorbed into a new constant $C_{n,p} > 0$, yielding the cleaner inequality

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^{p^*_{A_\infty}} w\right)^{\frac{1}{p^*_{A_\infty}}} \leq C_{n,p}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}.$$

However, we have chosen to keep the explicit dependence on the A_∞ constant in the statement to highlight that as $[w]_{A_\infty} \to 1$, the constant on the right-hand side goes to $c_n p^*$.

To put this result into perspective, it is worth noting that the weighted Poincaré inequality does not hold for the entire class A_∞, as shown in **[PR19, Theorem 1.26]**. Consequently, the Poincaré–Sobolev inequality does not hold either. In contrast, our result shows that if the A_∞ constant of w is not too large, that is, if w is not a highly degenerate weight, then a weighted Poincaré–Sobolev inequality can be established. In other words, the failure of a Poincaré inequality for the entire A_∞ class comes precisely from weights with large A_∞ constant, namely, the most singular weights in the class A_∞.

Chapter 4

Quantitative BMO–BLO estimates for the Hardy–Littlewood maximal function

In this chapter, we present the results of the following paper with some detailed proofs.

[Cla25b] Claros, A. *Quantitative BMO-BLO Estimates for the Hardy-Littlewood Maximal Function*, arXiv preprint arXiv:2511.18949 (2025).

Continuing the discussion of BMO, sharp maximal functions, and John–Nirenberg inequalities from Chapter 1, the aim of this chapter is to study quantitative versions of the classical mapping property of the Hardy–Littlewood maximal operator from BMO into BLO. We prove unweighted and weighted exponential estimates for the lower oscillation of $\mathcal{M}f$, normalized by the sharp maximal function $\mathcal{M}^{\#}f$, with linear dependence on the Fujii–Wilson constant $[w]_{A_\infty}$. Finally, we show that the A_∞ assumption is also necessary, obtaining a new quantitative characterization of the A_∞ class in terms of the boundedness of $\mathcal{M}$ from BMO into the weighted space BLO_w.

4.1 Introduction and motivation

As we reviewed in Chapter 1, the theory of functions of bounded mean oscillation (BMO) plays a fundamental role in modern harmonic analysis. The classical John–Nirenberg theorem (Theorem 1.7) states that functions of bounded mean oscillation are locally exponentially integrable. More precisely, if $f \in \mathrm{BMO}(\mathbb{R}^n)$, then there exist dimensional constants $c_1, c_2 > 0$ such that for every cube $Q \subset \mathbb{R}^n$,

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq c_1 e^{-c_2 t/\|f\|_{\mathrm{BMO}}}|Q|, \qquad t > 0.$$

This inequality establishes that bounded mean oscillation implies a form of exponential control stronger than any L^p bound. This property also identified the BMO space as a natural endpoint substitute for L^∞ in the scale of Lebesgue spaces and as a central object in the study of singular integrals, interpolation, and partial differential equations.

Many authors have studied quantitative versions of the John–Nirenberg inequality, including extensions to weighted settings **[PR19; OPRR20; MRR22; CP21]**, to metric measure spaces **[Str76; ST89; FPW00; DGL20; Myy22; GK23]**, and even to more general measures and capacities **[CS24; BCRS25]**. Indeed, the weighted John–Nirenberg, Poincaré, and Poincaré–Sobolev inequalities can be unified within a single theory through the so-called self-improving phenomena (see **[FPW00; MP02; PR19; GLP]**).

In **[Kar02]**, Karagulyan obtained an extension of the John–Nirenberg inequality, showing that one can recover the exponential decay of the oscillation of any locally integrable function once it is normalized by its sharp maximal function. Specifically, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a nonconstant function, then there exist dimensional constants $c_1, c_2 > 0$ such that for every cube Q,

$$\left|\left\{x \in Q : \frac{|f(x) - f_Q|}{\mathcal{M}^{\#} f(x)} > \lambda\right\}\right| \leq c_1 e^{-c_2 \lambda} |Q|,$$

for every $\lambda > 0$. To see why this constitutes a direct extension of the classical John–Nirenberg theorem, assume that $f \in \mathrm{BMO}(\mathbb{R}^n)$. As discussed in Chapter 1, the BMO seminorm of a function coincides with the L^∞ norm of its sharp maximal function. Thus, $\mathcal{M}^{\#} f(x) \leq \|f\|_{\mathrm{BMO}}$ for almost every $x \in \mathbb{R}^n$. Consequently, for any $t > 0$, the following inclusion of level sets holds up to a set of measure zero:

$$\left\{x \in Q : |f(x) - f_Q| > t\right\} \subset \left\{x \in Q : \frac{|f(x) - f_Q|}{\mathcal{M}^{\#} f(x)} > \frac{t}{\|f\|_{\mathrm{BMO}}}\right\}.$$

Applying Karagulyan's estimate to the set on the right-hand side with the parameter $\lambda = \frac{t}{\|f\|_{\mathrm{BMO}}}$ immediately yields the upper bound $c_1 e^{-\frac{c_2 t}{\|f\|_{\mathrm{BMO}}}} |Q|$, recovering the classical John–Nirenberg inequality.

This extension was later refined and generalized by Canto and Pérez **[CP21]**. Their result constitutes a twofold improvement over Karagulyan's estimate: first, the pointwise oscillation $|f(x) - f_Q|$ is replaced by the strictly larger local dyadic maximal operator $\mathcal{M}_Q(f - f_Q)$, and second, the estimate is extended to the weighted setting.

The precise statement is the following.

Theorem 4.1. *Let f be a nonconstant locally integrable function and let $w \in A_\infty$. Then, for any cube Q and any $1 \leq p < \infty$, the following estimate holds:*

$$\left(\frac{1}{w(Q)} \int_Q \left(\frac{\mathcal{M}_Q(f - f_Q)(x)}{\mathcal{M}^{\#} f(x)}\right)^p w(x)\, dx\right)^{\frac{1}{p}} \leq C_n [w]_{A_\infty} p, \tag{4.1}$$

where $\mathcal{M}_Q$ denotes the local dyadic maximal operator introduced in Chapter 1.

We also refer to **[LLO22]** for a different proof of Theorem 4.1. Furthermore, the linear dependence on the parameter p on the right-hand side of (4.1) is fundamental. By applying Proposition 4.4 with the associated probability measure $d\mu(x) = \frac{w(x)\, dx}{w(Q)}$, there exists a dimensional constant $c_n > 0$ such that

$$w\left(\left\{x \in Q : \frac{\mathcal{M}_Q(f - f_Q)(x)}{\mathcal{M}^{\#} f(x)} > t\right\}\right) \leq 2 e^{-\frac{c_n t}{[w]_{A_\infty}}} w(Q),$$

for every $t > 0$.

A natural variant of BMO is the space of bounded lower oscillation (BLO). Originally introduced by Coifman and Rochberg **[CR80]**, this space is defined by replacing the integral average f_Q with the essential infimum of the function over the cube.

Definition 4.2. A locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be of bounded lower oscillation, denoted by $f \in \mathrm{BLO}(\mathbb{R}^n)$, if

$$\|f\|_{\mathrm{BLO}} := \sup_Q \frac{1}{|Q|} \int_Q \big(f(x) - \operatorname{ess\,inf}_Q f\big)\, dx < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Unlike BMO, the space $\mathrm{BLO}(\mathbb{R}^n)$ is not a vector space, as it is not closed under multiplication by negative scalars (see for instance **[LYZ26, Proposition 4.1]**). However, it forms a convex positive cone. There is a direct relation between these spaces and the space $L^\infty(\mathbb{R}^n)$. For any $f \in L^\infty(\mathbb{R}^n)$ and any cube Q, it is clear that $f_Q - \operatorname{ess\,inf}_Q f \le 2\|f\|_{L^\infty}$, which trivially implies $L^\infty(\mathbb{R}^n) \subset \mathrm{BLO}(\mathbb{R}^n)$. Furthermore, since $f_Q - \operatorname{ess\,inf}_Q f \le \|f\|_{\mathrm{BLO}}$ for any $f \in \mathrm{BLO}(\mathbb{R}^n)$, a simple application of the triangle inequality shows the inclusion $\mathrm{BLO}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$. Hence, we have the following chain of inclusions:

$$L^\infty(\mathbb{R}^n) \subset \mathrm{BLO}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n).$$

Both inclusions are strict. On one hand, the function $f(x) = \max(0, -\log|x|)$ is an example of an unbounded function that belongs to $\mathrm{BLO}(\mathbb{R}^n)$, showing that $L^\infty \subsetneq \mathrm{BLO}$. On the other hand, its negative $f(x) = \min(0, \log|x|)$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$, but its essential infimum over any cube containing the origin is $-\infty$, meaning it falls outside of $\mathrm{BLO}(\mathbb{R}^n)$, which demonstrates $\mathrm{BLO} \subsetneq \mathrm{BMO}$ (see **[Ben82; LYZ26]**).

From the perspective of weight theory, these spaces admit a natural characterization in terms of logarithms of Muckenhoupt weights, as shown in **[CR80]**. In fact, one can write

$$\mathrm{BLO}(\mathbb{R}^n) = \{\alpha \log w : w \in A_1,\ \alpha \ge 0\},$$

and

$$\mathrm{BMO}(\mathbb{R}^n) = \{\alpha \log w : w \in A_\infty,\ \alpha \in \mathbb{R}\}.$$

This formulation highlights the intrinsic link between bounded oscillation spaces and Muckenhoupt weights, and also clearly reflects the strict inclusion $\mathrm{BLO} \subsetneq \mathrm{BMO}$.

In a classical result, Bennett, DeVore, and Sharpley **[BDS81]** proved that $\mathcal{M}$ maps BMO into itself. Shortly after, Bennett **[Ben82]** refined this result by proving that $\mathcal{M}$ actually maps BMO into the strictly smaller space BLO, satisfying

$$\|\mathcal{M}f\|_{\mathrm{BLO}} \le C_n \|f\|_{\mathrm{BMO}}, \tag{4.2}$$

for every function $f \in \mathrm{BMO}$ such that $\mathcal{M}f$ is not identically infinite.

The present chapter is devoted to proving a quantitative extension of the mapping property (4.2), analogous to (4.1). In particular, we obtain a fully quantitative version of this inequality in the spirit of the John–Nirenberg theorem, showing that the exponential decay of the lower oscillation of the maximal operator can be recovered for any locally integrable function when properly normalized by its sharp maximal function.

4.2 The unweighted case

To address the general weighted case, we first establish the corresponding estimate in the unweighted setting. Starting from the estimate for the local dyadic maximal function in Theorem 4.1, we can pass to the non-dyadic maximal operator, as in **[CP21, Theorem 5.1]**. Thus, for every nonconstant locally integrable function f and every cube Q,

$$\left|\left\{x \in Q : \frac{\mathcal{M}\left((f - f_Q)\chi_Q\right)(x)}{\mathcal{M}^{\#} f(x)} > t\right\}\right| \leq Ce^{-ct}|Q|, \tag{4.3}$$

for every $t > 0$, where $C, c > 0$ are dimensional constants. By the layer-cake formula and the estimate $\Gamma(p+1)^{1/p} \lesssim p$, this exponential decay implies

$$\left(\frac{1}{|Q|}\int_Q \left(\frac{\mathcal{M}((f - f_Q)\chi_Q)(x)}{\mathcal{M}^{\#} f(x)}\right)^p dx\right)^{\frac{1}{p}} \leq C_n p, \tag{4.4}$$

for every cube Q and every $1 \leq p < \infty$.

For completeness, and to keep the argument self-contained, we also give a direct proof of (4.4). The proof combines the unweighted case of Theorem 4.1 with Lemma 1.11.

Proof of (4.4). Fix a cube Q and $1 \leq p < \infty$. By Lemma 1.11, there exist $n+1$ dyadic grids $\{\mathcal{D}_k\}_{k=0}^n$ and corresponding cubes $Q_k \in \mathcal{D}_k$ such that $Q \subset \bigcap_{k=0}^n Q_k$ and $|Q_k| \leq c_n|Q|$ for all k. Furthermore, by (1.6) for any $x \in Q$, we have

$$\begin{aligned}
\mathcal{M}\big((f - f_Q)\chi_Q\big)(x) &\leq c_n \sum_{k=0}^n \mathcal{M}_d^{(k)}\big((f - f_Q)\chi_Q\big)(x) \\
&\leq c_n \sum_{k=0}^n \mathcal{M}_d^{(k)}\big((f - f_Q)\chi_{Q_k}\big)(x) \\
&= c_n \sum_{k=0}^n \mathcal{M}_{Q_k}\big(f - f_Q\big)(x),
\end{aligned}$$

where we have used $Q \subset Q_k$ and $Q_k \in \mathcal{D}_k$ for every $k \in \{0, \dots, n\}$. Let us analyze a single term in this sum for a fixed $k \in \{0, \dots, n\}$. By using $Q \subset Q_k$ and $|Q_k| \leq c_n|Q|$, we have

$$\begin{aligned}
\mathcal{M}_{Q_k}\big(f - f_Q\big)(x) &\leq \mathcal{M}_{Q_k}\big(f - f_{Q_k}\big)(x) + |f_{Q_k} - f_Q| \\
&\leq \mathcal{M}_{Q_k}\big(f - f_{Q_k}\big)(x) + \frac{1}{|Q|}\int_Q |f(y) - f_{Q_k}|\, dy \\
&\leq \mathcal{M}_{Q_k}\big(f - f_{Q_k}\big)(x) + \frac{|Q_k|}{|Q|}\frac{1}{|Q_k|}\int_{Q_k} |f(y) - f_{Q_k}|\, dy \\
&\leq \mathcal{M}_{Q_k}\big(f - f_{Q_k}\big)(x) + c_n \frac{1}{|Q_k|}\int_{Q_k} |f(y) - f_{Q_k}|\, dy \\
&\leq (1 + c_n)\mathcal{M}_{Q_k}\big(f - f_{Q_k}\big)(x),
\end{aligned}$$

for every $x \in Q$. Thus, we have proved

$$\mathcal{M}\big((f - f_Q)\chi_Q\big)(x) \leq c_n \sum_{k=0}^n \mathcal{M}_{Q_k}\big(f - f_{Q_k}\big)(x),$$

for every $x \in Q$. By applying Minkowski's inequality and Theorem 4.1 on each cube Q_k, we obtain

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q \left(\frac{\mathcal{M}((f-f_Q)\chi_Q)(x)}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} &\le c_n \left(\frac{1}{|Q|}\int_Q \left(\sum_{k=0}^{n}\frac{\mathcal{M}_{Q_k}(f-f_{Q_k})(x)}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} \\
&\le c_n \sum_{k=0}^{n}\left(\frac{1}{|Q|}\int_Q \left(\frac{\mathcal{M}_{Q_k}(f-f_{Q_k})(x)}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} \\
&\le c_n \sum_{k=0}^{n}\left(\frac{|Q_k|}{|Q|}\frac{1}{|Q_k|}\int_{Q_k} \left(\frac{\mathcal{M}_{Q_k}(f-f_{Q_k})(x)}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} \\
&\le c_n \sum_{k=0}^{n}\left(\frac{1}{|Q_k|}\int_{Q_k} \left(\frac{\mathcal{M}_{Q_k}(f-f_{Q_k})(x)}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} \\
&\le c_n \sum_{k=0}^{n} C_n\, p \\
&= c_n\, p.
\end{aligned}$$

This completes the proof. □

With this preliminary estimate established, we are now in a position to prove the unweighted version of our main result.

Proposition 4.3. *Let f be a nonconstant locally integrable function such that $\mathcal{M}f$ is not identically infinite. Then, there exists a dimensional constant $c_n > 0$ such that for every $1 \le p < \infty$, we have*

$$\left(\frac{1}{|Q|}\int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} \le c_n\, p,$$

for every cube Q.

Proof. Fix a cube Q and $1 \le p < \infty$. We decompose the maximal function as

$$\begin{aligned}
\mathcal{M}f(x) &= \mathcal{M}\big((f-f_{3Q})\chi_{3Q} + f_{3Q}\chi_{3Q} + f\chi_{(3Q)^c}\big)(x) \\
&\le \mathcal{M}\big((f-f_{3Q})\chi_{3Q}\big)(x) + \mathcal{M}g(x),
\end{aligned}$$

where $g(x) := f_{3Q}\chi_{3Q}(x) + f(x)\chi_{(3Q)^c}(x)$. Therefore,

$$\begin{aligned}
\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f &\le \mathcal{M}\big((f-f_{3Q})\chi_{3Q}\big)(x) + \mathcal{M}g(x) - \operatorname{ess\,inf}_Q \mathcal{M}f \\
&\le \mathcal{M}\big((f-f_{3Q})\chi_{3Q}\big)(x) + \big(\mathcal{M}g(x) - \operatorname{ess\,inf}_Q \mathcal{M}f\big)^{+}.
\end{aligned}$$

This yields the estimate

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} &\le \left(\frac{1}{|Q|}\int_Q \left(\frac{\mathcal{M}\big((f-f_{3Q})\chi_{3Q}\big)(x)}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} \\
&\quad + \left(\frac{1}{|Q|}\int_Q \left(\frac{\big(\mathcal{M}g(x) - \operatorname{ess\,inf}_Q \mathcal{M}f\big)^{+}}{\mathcal{M}^{\#}f(x)}\right)^p dx\right)^{\frac{1}{p}} \\
&=: I_1 + I_2.
\end{aligned}$$

By applying (4.4) over the larger cube $3Q$, we obtain

$$\begin{aligned} I_1 &\leq \left(\frac{1}{|Q|} \int_{3Q} \left(\frac{\mathcal{M}\big((f - f_{3Q})\chi_{3Q}\big)(x)}{\mathcal{M}^{\#} f(x)} \right)^p \, dx \right)^{\frac{1}{p}} \\ &\leq 3^{n/p} \left(\frac{1}{|3Q|} \int_{3Q} \left(\frac{\mathcal{M}\big((f - f_{3Q})\chi_{3Q}\big)(x)}{\mathcal{M}^{\#} f(x)} \right)^p \, dx \right)^{\frac{1}{p}} \\ &\leq c_n p. \end{aligned}$$

To estimate the second term, we bound $\mathcal{M}g$ pointwise in Q. Let $x \in Q$ and let P be a cube such that $x \in P$. We consider two cases. First, if $P \subseteq 3Q$, then

$$\frac{1}{|P|} \int_P |g(y)| \, dy = |f_{3Q}| \leq \frac{1}{|3Q|} \int_{3Q} |f(y)| \, dy \leq \operatorname{ess\,inf}_{z \in Q} \mathcal{M} f(z).$$

Otherwise, if $P \cap (3Q)^c \neq \emptyset$, we consider the smallest cube $\bar{P}$ that contains both P and $3Q$. Since $x \in Q \cap P$, we have $|\bar{P}| \leq c_n |P|$. Hence,

$$\begin{aligned} \frac{1}{|P|} \int_P |g(y)| \, dy &\leq \frac{1}{|P|} \int_P |g(y) - f_{\bar{P}}| \, dy + |f_{\bar{P}}| \\ &=: II_1 + II_2. \end{aligned}$$

We can estimate II_2 trivially,

$$II_2 \leq \frac{1}{|\bar{P}|} \int_{\bar{P}} |f(y)| \, dy \leq \operatorname{ess\,inf}_{z \in Q} \mathcal{M} f(z).$$

Using the definition of g, we have

$$\begin{aligned} II_1 &\leq c_n \frac{1}{|\bar{P}|} \int_{\bar{P}} |g(y) - f_{\bar{P}}| \, dy \\ &= c_n \frac{1}{|\bar{P}|} \int_{3Q} |f_{3Q} - f_{\bar{P}}| \, dy + c_n \frac{1}{|\bar{P}|} \int_{\bar{P} \cap (3Q)^c} |f(y) - f_{\bar{P}}| \, dy \\ &\leq c_n \frac{|3Q|}{|\bar{P}|} \frac{1}{|3Q|} \int_{3Q} |f(y) - f_{\bar{P}}| \, dy + c_n \frac{1}{|\bar{P}|} \int_{\bar{P} \cap (3Q)^c} |f(y) - f_{\bar{P}}| \, dy \\ &\leq c_n \frac{1}{|\bar{P}|} \int_{\bar{P}} |f(y) - f_{\bar{P}}| \, dy \\ &\leq c_n \mathcal{M}^{\#} f(x). \end{aligned}$$

We have proved that for each cube P containing x,

$$\frac{1}{|P|} \int_P |g(y)| \, dy \leq c_n \mathcal{M}^{\#} f(x) + \operatorname{ess\,inf}_{z \in Q} \mathcal{M} f(z),$$

and thus

$$\mathcal{M} g(x) \leq c_n \mathcal{M}^{\#} f(x) + \operatorname{ess\,inf}_{z \in Q} \mathcal{M} f(z),$$

for every $x \in Q$. Using this pointwise estimate, we can bound I_2,

$$I_2 \leq \left(\frac{1}{|Q|} \int_Q \left(\frac{c_n \mathcal{M}^{\#} f(x)}{\mathcal{M}^{\#} f(x)} \right)^p \, dx \right)^{\frac{1}{p}} = c_n \leq c_n p.$$

This concludes the proof. □

The linear dependence on p established in Proposition 4.3 yields exponential-type integrability. This equivalence is a consequence of the following result (see **[CP21]**).

Proposition 4.4. *Let (X, μ) be a probability space and let $f \geq 0$ be measurable. Assume that there exists a constant $\gamma > 0$ such that for every $p \geq 1$,*

$$\left(\int_X f(x)^p \, d\mu(x) \right)^{\frac{1}{p}} \leq \gamma \, p.$$

Then, we have $f \in \exp L(X, \mu)$, meaning

$$\mu(\{x \in X : \ f(x) > t\}) \leq 2 \exp\Big(- \frac{t}{4\,\gamma} \Big),$$

for every $t > 0$.

Proof. For each integer $k \geq 1$, the hypothesis gives

$$\int_X \left(\frac{f(x)}{4\gamma} \right)^k d\mu(x) \leq \left(\frac{k}{4} \right)^k.$$

Therefore, by the power series expansion of the exponential function,

$$\begin{aligned} \int_X \left(\exp\left(\frac{f(x)}{4\gamma} \right) - 1 \right) d\mu(x) &= \sum_{k=1}^{\infty} \frac{1}{k!} \int_X \left(\frac{f(x)}{4\gamma} \right)^k d\mu(x) \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{k}{4} \right)^k \\ &\leq 1. \end{aligned}$$

Now, for every $t > 0$, Chebyshev's inequality yields

$$\begin{aligned} \mu(\{x \in X : \ f(x) > t\}) &= \mu\left(\left\{ x \in X : \ \exp\left(\frac{f(x)}{4\gamma} \right) > \exp\left(\frac{t}{4\gamma} \right) \right\} \right) \\ &\leq \exp\left(-\frac{t}{4\gamma} \right) \int_X \exp\left(\frac{f(x)}{4\gamma} \right) d\mu(x) \\ &\leq 2 \exp\left(-\frac{t}{4\gamma} \right). \end{aligned}$$

□

We note that an alternative proof could also be obtained by following the same strategy as in Proposition 2.8; we have chosen to include this alternative technique.

As an immediate consequence of Proposition 4.3 and the previous result, we obtain the following corollary.

Corollary 4.5. *Let f be a nonconstant locally integrable function such that $\mathcal{M}f$ is not identically infinite. Then, there exists a dimensional constant $c_n > 0$ such that*

$$\left| \left\{ x \in Q : \frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^{\#} f(x)} > t \right\} \right| \leq 2e^{-c_n t} |Q|, \tag{4.5}$$

for all $t > 0$ and all cubes Q.

4.3 Weighted estimates and exponential decay

We now turn to the weighted setting. The main goal of this section is to prove the weighted extension of the BMO–BLO mapping property, establishing a precise linear dependence on the Fujii–Wilson A_∞ constant.

The first main result of the chapter is the following.

Theorem 4.6. *Let f be a nonconstant locally integrable function such that $\mathcal{M}f$ is not identically infinite, and let $w \in A_\infty$ be a weight. Then, for any cube Q and $1 \le p < \infty$, we have*

$$\left(\frac{1}{w(Q)}\int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f(x)}\right)^p w(x)\,dx\right)^{\frac{1}{p}} \le c_n\,[w]_{A_\infty}\,p. \tag{4.6}$$

In particular, if $f \in \mathrm{BMO}(\mathbb{R}^n)$, then for each $1 \le p < \infty$ we have

$$\left(\frac{1}{w(Q)}\int_Q \left(\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f\right)^p w(x)\,dx\right)^{\frac{1}{p}} \le c_n\,[w]_{A_\infty}\,p\,\|f\|_{\mathrm{BMO}}, \tag{4.7}$$

for all cubes Q.

Remark 4.7. The assumption that $\mathcal{M}f$ is not identically infinite cannot be removed. In **[BDS81]**, it is shown that this condition is necessary for the validity of the boundedness of $\mathcal{M}$ on BMO, and an explicit counterexample is provided by the function $f(x) = \log|x|$. We also refer to the recent work **[BGKM25, Appendix A]**, which characterizes the class of measures μ for which $\mathcal{M}\mu$ is finite almost everywhere.

The linear dependence on p in the right-hand side of (4.6) implies the corresponding exponential estimates. We formalize this exponential decay in the following proposition.

Proposition 4.8. *Let f be a nonconstant locally integrable function such that $\mathcal{M}f$ is not identically infinite, and let $w \in A_\infty$ be a weight. There exist dimensional constants $c_1, c_2 > 0$ such that*

$$w\left(\left\{x \in Q : \frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f(x)} > t\right\}\right) \le c_1 e^{-\frac{c_2 t}{[w]_{A_\infty}}} w(Q), \tag{4.8}$$

for all $t > 0$ and all cubes Q. That is,

$$\left\|\frac{\mathcal{M}f - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f}\right\|_{\exp L\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} \le c_n [w]_{A_\infty}.$$

To prove these results, we have chosen to follow the strategy of first establishing the unweighted case and then passing to the weighted estimate. One might naturally wonder if the direct proof of Proposition 4.3 could be adapted to the weighted setting. While the local and non-local decomposition of the maximal function and the corresponding pointwise estimates remain completely valid, a direct adaptation introduces the ratio $\left(\frac{w(3Q)}{w(Q)}\right)^{1/p}$ when bounding the local part over the enlarged cube $3Q$. Controlling this ratio requires the use of the doubling property of A_∞ weights. However, as shown in Theorem 1.20 (see also Remark 1.21), the available quantitative control of the doubling constant for an arbitrary weight $w \in A_\infty$ has a dependence on the Fujii–Wilson constant that is worse than exponential.

Consequently, our approach of passing through the unweighted exponential decay circumvents this loss and yields the sharp linear dependence on $[w]_{A_\infty}$ in (4.6).

The bridge between the Lebesgue measure and the weighted setting is provided by the following property of A_∞ weights, which is a simple consequence of the reverse Hölder inequality (see **[HPR12]**). A weight $w \in A_\infty$ satisfies

$$\frac{w(E)}{w(Q)} \leq 2\left(\frac{|E|}{|Q|}\right)^{\frac{1}{2^{n+1}[w]_{A_\infty}}},$$

for every cube Q and every measurable set $E \subseteq Q$. Using this property and the unweighted exponential bound (4.5), we can easily prove Proposition 4.8. In fact, the L^p estimate in Theorem 4.6 and the exponential decay (4.8) are equivalent: the exponential estimate implies the L^p bound via the layer-cake representation, and conversely, the linear L^p growth directly yields the exponential decay via Proposition 4.4. We present the proof in the direction from the exponential decay to the L^p bound.

Proof of Proposition 4.8. Let $w \in A_\infty$ and fix a cube Q. Let $t > 0$ and denote the relevant level set by

$$E_t = \left\{x \in Q : \frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f(x)} > t\right\}.$$

By the aforementioned property of A_∞ weights and the unweighted exponential estimate (4.5) obtained in Corollary 4.5, we have

$$\begin{aligned} w(E_t) &\leq 2\left(\frac{|E_t|}{|Q|}\right)^{\frac{1}{2^{n+1}[w]_{A_\infty}}} w(Q) \\ &\leq 2\left(2e^{-c_n t}\right)^{\frac{1}{2^{n+1}[w]_{A_\infty}}} w(Q) \\ &= 2^{1+\frac{1}{2^{n+1}}} e^{\frac{-c_n t}{2^{n+1}[w]_{A_\infty}}} w(Q). \end{aligned}$$

This concludes the proof of (4.8). □

As an immediate consequence of Proposition 4.8, we obtain the following good-lambda inequality.

Corollary 4.9. *Let f be a nonconstant locally integrable function such that $\mathcal{M}f$ is not identically infinite, and let $w \in A_\infty$ be a weight. There exist dimensional constants $c_1, c_2 > 0$ such that for every cube Q and for every $\lambda, \gamma > 0$ the following good-lambda inequality holds:*

$$w\left(\left\{x \in Q : \mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f > \lambda, \mathcal{M}^\# f(x) \leq \gamma\lambda\right\}\right) \leq c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} w(Q). \tag{4.9}$$

Remark 4.10. We note that inequality (4.9) is an improvement of the classical good-lambda estimate of Journé **[Jou83, p. 43]**, since it provides exponential decay.

Proof of Corollary 4.9. Fix $\lambda, \gamma > 0$ and consider any point $x \in Q$ satisfying both $\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f > \lambda$ and $\mathcal{M}^\# f(x) \leq \gamma\lambda$. For any such x, we have

$$\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f(x)} > \frac{\lambda}{\mathcal{M}^\# f(x)} \geq \frac{\lambda}{\gamma\lambda} = \frac{1}{\gamma}.$$

This pointwise bound implies the set inclusion

$$\left\{x \in Q : \mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f > \lambda, \mathcal{M}^\# f(x) \leq \gamma\lambda\right\} \subseteq E_{\frac{1}{\gamma}}.$$

Using this inclusion, and applying the exponential estimate (4.8) that we just proved evaluated at $t = 1/\gamma$, we conclude that

$$w\left(\left\{x \in Q : \mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f > \lambda, \mathcal{M}^{\#}f(x) \le \gamma\lambda\right\}\right) \le w\left(E_{\frac{1}{\gamma}}\right) \le c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} w(Q).$$

This establishes (4.9) and completes the proof. □

Proof of Theorem 4.6. Let $1 \le p < \infty$, $w \in A_\infty$ and let Q be a cube. For each $t > 0$, define the level set

$$E_t = \left\{x \in Q : \frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^{\#}f(x)} > t\right\}.$$

By the layer-cake formula and the weak weighted estimate (4.8) established in Proposition 4.8, we have

$$\begin{aligned}\left(\frac{1}{w(Q)}\int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^{\#}f(x)}\right)^p w(x)\,dx\right)^{\frac{1}{p}} &= \left(p\int_0^\infty t^{p-1}\frac{w(E_t)}{w(Q)}\,dt\right)^{\frac{1}{p}} \\ &\le \left(c_1 p\int_0^\infty t^{p-1} e^{-\frac{c_2 t}{[w]_{A_\infty}}}\,dt\right)^{\frac{1}{p}} \\ &= c_1^{\frac{1}{p}}\frac{[w]_{A_\infty}}{c_2}\left(p\int_0^\infty s^{p-1}e^{-s}\,ds\right)^{\frac{1}{p}} \\ &= c_1^{\frac{1}{p}}\frac{[w]_{A_\infty}}{c_2}\left(p\Gamma(p)\right)^{\frac{1}{p}} \\ &\le \frac{c_1}{c_2}[w]_{A_\infty}\Gamma(p+1)^{\frac{1}{p}},\end{aligned}$$

where we applied the change of variables $s = \frac{c_2 t}{[w]_{A_\infty}}$. We obtain the desired estimate by noting that $\Gamma(p+1)^{\frac{1}{p}} \lesssim p$ (see for instance **[Gra24, Corollary 6.2.5]**). To prove the second part, assume that $f \in \mathrm{BMO}$. Since $\mathcal{M}^{\#}f(x) \le \|f\|_{\mathrm{BMO}}$ for almost every x, we have

$$\begin{aligned}\left(\frac{1}{w(Q)}\int_Q \left(\frac{\mathcal{M}f - \operatorname{ess\,inf}_Q \mathcal{M}f}{\|f\|_{\mathrm{BMO}}}\right)^p w\,dx\right)^{\frac{1}{p}} &\le \left(\frac{1}{w(Q)}\int_Q \left(\frac{\mathcal{M}f - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^{\#}f}\right)^p w\,dx\right)^{\frac{1}{p}} \\ &\le c_n[w]_{A_\infty}\,p.\end{aligned}$$

Multiplying both sides by $\|f\|_{\mathrm{BMO}}$ yields (4.7) and concludes the proof. □

Remark 4.11. Using similar arguments, Theorem 4.6 can be extended to the Lorentz scale $L^{p,q}$. For any $1 \le p, q < \infty$, applying the exponential decay (4.8) directly yields

$$\begin{aligned}\left\|\frac{\mathcal{M}f - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^{\#}f}\right\|_{L^{p,q}\left(Q, \frac{w(x)\,dx}{w(Q)}\right)} &\le \left(p\int_0^\infty t^{q-1}\left(c_1 e^{-\frac{c_2 t}{[w]_{A_\infty}}}\right)^{\frac{q}{p}}\,dt\right)^{\frac{1}{q}} \\ &= C_n[w]_{A_\infty}\frac{p}{q}\left(p\Gamma(q)\right)^{\frac{1}{q}},\end{aligned}$$

for every cube Q.

4.4 A new quantitative characterization of A_∞

The results obtained in the previous sections establish that the A_∞ condition is sufficient to guarantee the boundedness of the maximal operator $\mathcal{M}$ from BMO into the weighted space BLO_w. Specifically, the previous inequality (4.7), in the particular case $p = 1$, tells us that if $w \in A_\infty$, then we have the boundedness

$$\|\mathcal{M}f\|_{\mathrm{BLO}_w} \leq C_n [w]_{A_\infty} \|f\|_{\mathrm{BMO}},$$

where the weighted BLO_w seminorm is defined by

$$\|f\|_{\mathrm{BLO}_w} := \sup_Q \frac{1}{w(Q)} \int_Q \left(f(x) - \operatorname{ess\,inf}_Q f\right) w(x)\, dx.$$

A natural question arises: is the A_∞ condition also necessary for such an estimate to hold? We shall prove that the answer is affirmative. In fact, the boundedness of the Hardy–Littlewood maximal operator from BMO into BLO_w is not only a consequence of the A_∞ condition, but actually a defining property of the class. This equivalence provides a new quantitative characterization of A_∞ in terms of the boundedness of the maximal operator $\mathcal{M}$ from BMO into BLO_w.

Theorem 4.12. *Let w be a weight. Then there exist some dimensional constants c_n, C_n such that*

$$c_n [w]_{A_\infty} \leq \sup_f \|\mathcal{M}f\|_{\mathrm{BLO}_w} \leq C_n [w]_{A_\infty}, \tag{4.10}$$

where the supremum is taken over all functions f such that $\|f\|_{\mathrm{BMO}} = 1$ and $\mathcal{M}f$ is not identically infinite. As a consequence, we have the following quantitative equivalence of the Fujii–Wilson A_∞ constant,

$$[w]_{A_\infty} \simeq_n \sup_f \sup_Q \frac{1}{w(Q)} \int_Q \left(\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f\right) w(x)\, dx.$$

As a direct corollary, we obtain that the class A_∞ is both necessary and sufficient for the validity of our main result, inequality (4.6).

Corollary 4.13. *Let w be a weight.*

- *Sufficiency: Let f be a nonconstant locally integrable function such that $\mathcal{M}f$ is not identically infinite. If $w \in A_\infty$, then for any cube Q and $1 \leq p < \infty$, we have*

$$\left(\frac{1}{w(Q)} \int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f(x)}\right)^p w(x)\, dx\right)^{\frac{1}{p}} \leq c_n [w]_{A_\infty} p.$$

- *Necessity: Assume that the estimate*

$$\left(\frac{1}{w(Q)} \int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f(x)}\right)^p w(x)\, dx\right)^{\frac{1}{p}} \leq C\, p \tag{4.11}$$

holds for every nonconstant locally integrable function f such that $\mathcal{M}f$ is not identically infinite, every $1 \leq p < \infty$, and every cube Q. Then $w \in A_\infty$ and $[w]_{A_\infty} \lesssim_n C$.

The remainder of this section is devoted to the proof of Theorem 4.12 and its Corollary 4.13. Within the proof of the Theorem, we will address some technical difficulties caused by the maximal operator, and then follow the lines of the proof of Theorem 1.2 in **[OPRR20]**.

Proof of Theorem 4.12. In (4.7), with $p = 1$, we have already proved that

$$\|\mathcal{M}f\|_{\mathrm{BLO}_w} \leq c_n [w]_{A_\infty} \|f\|_{\mathrm{BMO}},$$

which provides the second inequality in (4.10). To prove the first inequality, we define

$$X = \sup_{\|f\|_{\mathrm{BMO}}=1} \|\mathcal{M}f\|_{\mathrm{BLO}_w},$$

where the supremum is taken over all functions f such that $\|f\|_{\mathrm{BMO}} = 1$ and $\mathcal{M}f$ is not identically infinite. We have to prove $[w]_{A_\infty} \lesssim_n X$. Without loss of generality, we can assume that $X < \infty$.

Let Q be a cube. We define the auxiliary weight

$$v := \left[\mathcal{M}\left(\frac{w\chi_Q}{w_Q}\right)\right]^{\frac{1}{2}}.$$

It is well known that $v \in A_1$ with $[v]_{A_1} \leq c_n$ (see for instance **[GF85]**). Let us define $b := \log^+ v = \max(0, \log v)$, which satisfies $b \in \mathrm{BMO}$ and $\|b\|_{\mathrm{BMO}} \leq \frac{3}{2}\|\log v\|_{\mathrm{BMO}} \leq \rho_n$ (see **[Gra24, Proposition 6.1.4]**). Using the definition of X, we obtain

$$\|\mathcal{M}b\|_{\mathrm{BLO}_w} \leq \|b\|_{\mathrm{BMO}} X \leq \rho_n X,$$

and hence,

$$\frac{1}{w(Q)} \int_Q \left(\mathcal{M}b(x) - \operatorname{ess\,inf}_Q \mathcal{M}b\right) w(x)\, dx \leq \rho_n X. \tag{4.12}$$

We now estimate the essential infimum term. Fix $x \in Q$, and let P be any cube with $x \in P$. By Jensen's inequality,

$$\begin{aligned}
\frac{1}{|P|}\int_P |b(y)|\, dy = \frac{1}{|P|}\int_P \log^+ v(y)\, dy &\leq \frac{1}{|P|}\int_P \log(v(y)+1)\, dy \\
&\leq \log\left(\frac{1}{|P|}\int_P v(y)\, dy + 1\right) \\
&\leq \log\left(\mathcal{M}v(x) + 1\right).
\end{aligned}$$

Consequently, we have $\mathcal{M}b(x) \leq \log\left(\mathcal{M}v(x) + 1\right)$ for every $x \in Q$. Using the A_1 condition and that $v \geq 1$ in Q, we have

$$\mathcal{M}b(x) \leq \log([v]_{A_1} v(x) + 1) \leq \log([v]_{A_1} + 1) + \log v(x) \leq \log c_n + \log v(x),$$

for almost every $x \in Q$. Therefore,

$$\operatorname{ess\,inf}_{z\in Q} \mathcal{M}b(z) \leq \log c_n + \operatorname{ess\,inf}_{z\in Q} \log(v(z)).$$

Using the Kolmogorov inequality together with the weak $(1,1)$ boundedness of the maximal operator (see **[Gra14, p. 100]**), we obtain

$$\begin{aligned}
\operatorname{ess\,inf}_{z\in Q} \mathcal{M}b(z) &\leq \log c_n + \operatorname{ess\,inf}_{z\in Q} \log(v(z)) \\
&\leq \log c_n + \frac{1}{|Q|}\int_Q \log v \\
&\leq \log c_n + \log\left(\frac{1}{|Q|}\int_Q v\right) \\
&= \log c_n + \log\left(\frac{1}{|Q|}\int_Q \mathcal{M}\left(\frac{w\chi_Q}{w_Q}\right)^{\frac{1}{2}}\right) \\
&\leq \log c_n + \log\left(2\left\|\mathcal{M}\left(\frac{w\chi_Q}{w_Q}\right)\right\|^{\frac{1}{2}}_{L^{1,\infty}(\mathbb{R}^n,\frac{1}{|Q|})}\right) \\
&\leq \log c_n + \log c_n' \\
&= \log \alpha_n,
\end{aligned}$$

where $\alpha_n := c_n c_n'$. Now, note that

$$\frac{1}{|Q|}\int_Q \log^+\left(\frac{w(x)}{\alpha_n^2 w_Q}\right) w(x)\,dx = \frac{1}{|Q|}\int_{L_Q} \log^+\left(\frac{w(x)}{\alpha_n^2 w_Q}\right) w(x)\,dx,$$

where $L_Q = \{x \in Q : w(x) \geq \alpha_n^2 w_Q\}$. Hence, for a.e. $x \in L_Q$ we have

$$\begin{aligned}
\mathcal{M}b(x) - \operatorname{ess\,inf}_{z\in Q} \mathcal{M}b(z) &\geq \mathcal{M}b(x) - \log \alpha_n \\
&\geq b(x) - \log \alpha_n \\
&= \log v(x) - \log \alpha_n \\
&= \frac{1}{2}\log\left(\frac{1}{\alpha_n^2}\mathcal{M}\left(\frac{w\chi_Q}{w_Q}\right)(x)\right) \\
&\geq \frac{1}{2}\log\left(\frac{1}{\alpha_n^2}\frac{w(x)}{w_Q}\right) \\
&= \frac{1}{2}\log^+\left(\frac{w(x)}{\alpha_n^2 w_Q}\right).
\end{aligned}$$

Combining this with (4.12), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{1}{|Q|}\int_Q \log^+\left(\frac{w(x)}{\alpha_n^2 w_Q}\right) w(x)\,dx &= \frac{1}{|Q|}\int_{L_Q} \frac{1}{2}\log^+\left(\frac{w(x)}{\alpha_n^2 w_Q}\right) w(x)\,dx \\
&\leq \frac{1}{|Q|}\int_{L_Q} \left(\mathcal{M}b(x) - \operatorname{ess\,inf}_{z\in Q} \mathcal{M}b(z)\right) w(x)\,dx \\
&\leq \frac{1}{|Q|}\int_{Q} \left(\mathcal{M}b(x) - \operatorname{ess\,inf}_{z\in Q} \mathcal{M}b(z)\right) w(x)\,dx \\
&\leq \frac{w(Q)}{|Q|}\rho_n X.
\end{aligned}$$

We conclude that

$$\begin{aligned}\frac{1}{|Q|}\int_Q \mathcal{M}(w\chi_Q)(x)\,dx &\simeq_n \frac{1}{|Q|}\int_Q \left(1+\log^+\left(\frac{w(x)}{w_Q}\right)\right) w(x)\,dx\\ &= \frac{w(Q)}{|Q|}+\frac{1}{|Q|}\int_Q \log^+\left(\alpha_n^2\frac{w(x)}{\alpha_n^2 w_Q}\right) w(x)\,dx\\ &\le 3\log\alpha_n\frac{w(Q)}{|Q|}+\frac{1}{|Q|}\int_Q \log^+\left(\frac{w(x)}{\alpha_n^2 w_Q}\right) w(x)\,dx\\ &\le 3\log\alpha_n\frac{w(Q)}{|Q|}+2\frac{w(Q)}{|Q|}\rho_n X\\ &\le C_n\frac{w(Q)}{|Q|}X,\end{aligned}$$

where we refer to **[HP13, Lemma 6.2]** and **[Ste69]** for the first equivalence (see also Remark 3.31), and in the last inequality we have used the bound $X \ge c_n$, which remains to be proved.

We now establish the inequality $X \ge c_n$. Let $Q = [0,1]^n$ be the unit cube and decompose it into its 2^n dyadic children. Let E be one of them satisfying $w(E) = \max\{w(Q_j) : Q_j \text{ is a dyadic child of } Q\}$. By the additivity of the measure w, we have $w(E) \ge 2^{-n}w(Q)$. Let $f := 2\chi_E$, then we have $\|\chi_E\|_{\mathrm{BMO}} = \frac{1}{2}$ (see **[BE20]**), and thus $\|f\|_{\mathrm{BMO}} = 1$. For every $x \in Q$ we have

$$\mathcal{M}f(x) \ge \frac{1}{|Q|}\int_Q f(y)\,dy = \frac{2|E|}{|Q|} = 2^{1-n}.$$

Therefore, we have proved $\operatorname{ess\,inf}_{x\in Q}\mathcal{M}f(x) \ge 2^{1-n}$.

To prove the reverse inequality, let z_0 be the vertex of Q strictly opposite to E. That is, the distance in the supremum norm between z_0 and E is exactly $\frac{1}{2}$. Let P be any cube of side length L and such that $z_0 \in P$. If $L \le \frac{1}{2}$, the interior of P cannot intersect E, and the average of f over P is 0. If $L > \frac{1}{2}$, the measure of the intersection is bounded by $(L-\frac{1}{2})^n$. Hence,

$$\frac{1}{|P|}\int_P f(y)\,dy = 2\frac{|P\cap E|}{|P|} \le 2\frac{(L-\frac{1}{2})^n}{L^n} = 2\left(1-\frac{1}{2L}\right)^n.$$

This expression is strictly increasing with respect to L. For $L \le 1$, its maximum value is attained at $L = 1$, yielding $2(1-\frac{1}{2})^n = 2^{1-n}$. For $L > 1$, the measure of the intersection is bounded by $|E| = (\frac{1}{2})^n$, and then the average is strictly less than $2(\frac{1}{2})^n = 2^{1-n}$. Taking the supremum over all cubes P such that $z_0 \in P$, we obtain $\mathcal{M}f(z_0) \le 2^{1-n}$. Since f is continuous in a neighborhood of z_0, the maximal function $\mathcal{M}f$ is continuous at z_0 (see for instance **[Kur15; WY20]**). Since z_0 is a vertex of Q, this continuity implies that for every $\varepsilon > 0$, the set $\{x \in Q : \mathcal{M}f(x) < 2^{1-n}+\varepsilon\}$ contains the intersection of Q with a neighborhood of z_0, which has strictly positive Lebesgue measure because z_0 is a vertex of Q. This implies that $\operatorname{ess\,inf}_{x\in Q}\mathcal{M}f(x) \le 2^{1-n}$, and thus $\operatorname{ess\,inf}_{x\in Q}\mathcal{M}f(x) = 2^{1-n}$.

We can now estimate the BLO_w seminorm of $\mathcal{M}f$,

$$\begin{aligned}\frac{1}{w(Q)}\int_Q \left(\mathcal{M}f - \operatorname{ess\,inf}_Q \mathcal{M}f\right) w(x)\,dx &\ge \frac{1}{w(Q)}\int_E \left(\mathcal{M}f - \operatorname{ess\,inf}_Q \mathcal{M}f\right) w(x)\,dx\\ &= \frac{1}{w(Q)}\int_E \left(2-2^{1-n}\right) w(x)\,dx\end{aligned}$$

$$= (2 - 2^{1-n})\frac{w(E)}{w(Q)}$$
$$\geq (2 - 2^{1-n})2^{-n}.$$

This proves that $X \geq c_n$ with $c_n := (2 - 2^{1-n})2^{-n} > 0$ and completes the proof of the theorem. □

We will now apply this characterization of the A_∞ class to prove Corollary 4.13, which follows as a direct consequence of the quantitative inequalities established above.

Proof of Corollary 4.13. The sufficiency is the content of Theorem 4.6. To prove the necessity, let w be a weight and assume that inequality (4.11) holds. In particular, if $f \in \mathrm{BMO}(\mathbb{R}^n)$, then taking $p = 1$ in (4.11) and using $\mathcal{M}^\# f \leq \|f\|_{\mathrm{BMO}}$ we have

$$\|\mathcal{M}f\|_{\mathrm{BLO}_w} \leq C\|f\|_{\mathrm{BMO}}.$$

By Theorem 4.12, this yields $w \in A_\infty$. In addition, by taking the supremum over f we obtain

$$[w]_{A_\infty} \simeq_n \sup_f \sup_Q \frac{1}{w(Q)} \int_Q \left(\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f\right) w(x)\, dx \leq C.$$

This completes the proof. □

Remark 4.14. A similar characterization holds for the main result of **[CP21]**, namely inequality (4.1). The proof follows exactly the same argument as in the corollary above, replacing the use of Theorem 4.12 with **[OPRR20, Theorem 1.2]**.

Remark 4.15. As a concluding observation, Corollary 4.13 provides the following quantitative characterization

$$[w]_{A_\infty} \simeq_n \sup_f \sup_{p\geq 1} \sup_Q \frac{1}{p} \left(\frac{1}{w(Q)} \int_Q \left(\frac{\mathcal{M}f(x) - \operatorname{ess\,inf}_Q \mathcal{M}f}{\mathcal{M}^\# f(x)}\right)^p w(x)\, dx\right)^{\frac{1}{p}},$$

where the first supremum is taken over all nonconstant locally integrable functions f such that $\mathcal{M}f$ is not identically infinite.

Chapter 5

Bourgain–Brezis–Mironescu formula for Riesz potentials

Here, we present the results of the following work.

[CP26] Claros, A. and Pérez, C. *Bourgain–Brezis–Mironescu formula for Riesz potentials*, arXiv preprint arXiv:2604.06827 (2026).

The purpose of this chapter is to connect the Bourgain–Brezis–Mironescu limiting phenomenon with fractional subrepresentation formulas involving Riesz potentials. More precisely, we study the nonlinear fractional differential operator $\mathcal{D}^\alpha f$ and identify the pointwise limit of $(1-\alpha)I_\alpha(\mathcal{D}^\alpha f)$ when $\alpha \to 1^-$. The main result of this chapter shows that this operator converges pointwise to the first-order potential $K_n I_1(|\nabla f|)$. We also discuss the extension of this convergence for Sobolev functions and include a natural extension to the L^p-scale of fractional differential operators.

5.1 Introduction and statement of the main result

A classical starting point in Sobolev theory is the subrepresentation formula

$$|f(x)| \lesssim_n I_1(|\nabla f|)(x), \tag{5.1}$$

valid for $f \in C_c^\infty(\mathbb{R}^n)$, with $n \geq 2$ (see Remark 1.29). For $\alpha \in (0,n)$ we consider the Riesz potential

$$I_\alpha f(x) = \gamma_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

where $\gamma_{n,\alpha}$ is defined by

$$\gamma_{n,\alpha} = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)}. \tag{5.2}$$

The estimate (5.1) can be derived from the classical local $(1,1)$-Poincaré inequality on balls (or cubes) via a standard chain argument stated in Lemma 1.31, which converts local mean oscillation control into a pointwise potential estimate. Recently, several extensions of (5.1) have been obtained.

On the one hand, the authors in **[HMP25b]** refine the left-hand side through the substitution $f \mapsto Tf$, with T ranging from various maximal operators to singular integrals with rough kernels. On the other hand, **[HMP25a]** improves the right-hand side by replacing it with a smaller operator (see (5.8) below). This improvement of (5.1) is closely related to the very interesting nonlinear fractional differential operator introduced by Spector in **[Spe20]**:

$$\mathcal{D}^\alpha f(x) := \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+\alpha}}\, dy, \qquad 0 < \alpha < 1,$$

which serves as a nonlocal "fractional gradient" of order α. This operator is of interest for several reasons, one of them is the recent fractional subrepresentation formula established in **[HMP25a]** (see (5.6) below). On the other hand, it preserves some of the structural properties of first-order differential operators. For instance, it satisfies the following Leibniz-type inequality (see **[NP09]** and Proposition 5.11 below),

$$\|\mathcal{D}^\alpha(fg)\|_{L^1(\mathbb{R}^n)} \le \|f\, \mathcal{D}^\alpha g\|_{L^1(\mathbb{R}^n)} + \|g\, \mathcal{D}^\alpha f\|_{L^1(\mathbb{R}^n)}.$$

The link between the fractional scale and the gradient scale is given by the Bourgain–Brezis–Mironescu (BBM) formula **[BBM01]**[1]. For $f \in C_c^\infty(\mathbb{R}^n)$, it identifies the limit of the fractional Gagliardo seminorm

$$\lim_{\alpha \to 1^-} (1-\alpha) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+\alpha}}\, dx\, dy = K_n \int_{\mathbb{R}^n} |\nabla f(x)|\, dx, \tag{5.3}$$

where

$$K_n := \int_{\mathbb{S}^{n-1}} |\omega \cdot e|\, d\sigma(\omega) \tag{5.4}$$

and $e \in \mathbb{S}^{n-1}$ is arbitrary. In particular, the factor $(1-\alpha)$ is crucial, otherwise the limit is not even finite for nonconstant functions (see **[Bre02]**). BBM-type limits have also been investigated in extension domains **[BMR20]**, in arbitrary bounded domains, where boundary effects require modifications of the inner integral in the seminorm **[DD22]**, and in fully arbitrary domains with $W_q^{\alpha,p}$ seminorms **[Moh24]**, which are related to Triebel–Lizorkin spaces. Further extensions have been obtained in fractional Orlicz–Sobolev spaces **[ACPS20; ACPS21a; ACPS21b]**, in the setting of interpolation spaces **[DM23]**, in ball Banach function spaces **[DGPYYZ24; ZYY24a; CGYY25; LYYZZ26]**, in connection with Triebel–Lizorkin spaces **[BSY23]**, and for anisotropic fractional energies **[FS25]**.

A complementary endpoint perspective was introduced by Brezis, Van Schaftingen, and Yung **[BVY21a]**, who showed that the Sobolev seminorm can also be recovered by replacing the strong L^p norm in the endpoint Gagliardo seminorm by a weak L^p quasi-norm. This point of view led to Lorentz-space substitutes for fractional Sobolev and Gagliardo–Nirenberg estimates in limiting cases **[BVY21b]**, and was further developed by Brezis, Seeger, Van Schaftingen, and Yung through families of functionals based on superlevel sets of difference quotients **[BSVY21; BSVY22]**. Related extensions in ball Banach function and Sobolev spaces were obtained in **[DLYYZ23; ZYY23; ZYY24b]**; see also **[HLYYZ25]** for higher-order analogues in ball Banach function spaces. We also refer to **[DLYYZ22]** for related characterizations on spaces of homogeneous type under suitable Poincaré inequalities.

[1]The original proof of the BBM formula was stated and proved in **[BBM01]** for smooth bounded domains $\Omega \subset \mathbb{R}^n$. The result can be extended to the whole $\mathbb{R}^n$, see **[BSY23, Appendix A]** and **[Moh24]**.

In **[BBM02]**, the following fractional Poincaré inequality on cubes is proved with the BBM extra term $(1-\alpha)$. For every cube $Q \subset \mathbb{R}^n$, every $0 < \alpha < 1$ and every $f \in C^\infty(Q)$,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \lesssim (1-\alpha)\,\ell(Q)^\alpha\,\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dx\,dy. \tag{5.5}$$

Once (5.5) is available, a standard telescoping argument (see Lemma 1.31) yields the pointwise fractional subrepresentation formula

$$|f(x)| \lesssim_n (1-\alpha)\,I_\alpha(\mathcal{D}^\alpha f)(x), \qquad x \in \mathbb{R}^n, \quad f \in C_c^\infty(\mathbb{R}^n), \tag{5.6}$$

as shown in **[HMP25a]**. In particular, $(1-\alpha)I_\alpha(\mathcal{D}^\alpha f)$ plays the role of a fractional version of the first-order Riesz potential of the gradient, in the spirit of (5.1). Interesting related subrepresentation formulas were established for s-John domains in **[DD18]**, although without the corresponding BBM-type factor $(1-\alpha)$.

An alternative motivation for studying $(1-\alpha)I_\alpha(\mathcal{D}^\alpha f)$ stems from the theory of rough singular integrals. In the recent work **[HMP25a]**, it is established that, for every $\alpha \in (0,1)$, if Ω is homogeneous of degree zero, has vanishing average on $\mathbb{S}^{n-1}$, and belongs to the Marcinkiewicz space $L^{\frac{n}{\alpha},\infty}(\mathbb{S}^{n-1})$, then the maximal truncated rough operator

$$T_\Omega^\star f(x) := \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n}\,f(y)\,dy\right|$$

satisfies the pointwise estimate

$$T_\Omega^\star f(x) \le c_n\,\|\Omega\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{S}^{n-1})}\,(1-\alpha)\,I_\alpha(\mathcal{D}^\alpha f)(x), \qquad f \in C_c^\infty(\mathbb{R}^n). \tag{5.7}$$

The same work also establishes that for every $f \in C_c^\infty(\mathbb{R}^n)$,

$$(1-\alpha)\,I_\alpha(\mathcal{D}^\alpha f)(x) \le C_{n,\alpha}\,I_1(|\nabla f|)(x), \tag{5.8}$$

where the constant $C_{n,\alpha}$ remains bounded as $\alpha \to 1^-$ (for completeness, we include a proof in Theorem 5.3 below). Combining this with (5.6), we obtain

$$|f(x)| \lesssim_n (1-\alpha)\,I_\alpha(\mathcal{D}^\alpha f)(x) \le C_{n,\alpha}\,I_1(|\nabla f|)(x),$$

for every $x \in \mathbb{R}^n$. The boundedness of $C_{n,\alpha}$ when $\alpha \to 1^-$ in (5.8) suggests that the operator $(1-\alpha)I_\alpha(\mathcal{D}^\alpha f)$ should have a meaningful limit as $\alpha \to 1^-$, in the spirit of (5.3).

Our main result in this chapter confirms this intuition by identifying the pointwise BBM limit of the operator appearing on the left-hand side of (5.8).

Theorem 5.1. *Let $f \in C_c^\infty(\mathbb{R}^n)$ with $n \ge 2$. Then, for every $x \in \mathbb{R}^n$,*

$$\lim_{\alpha\to 1^-}(1-\alpha)\,I_\alpha(\mathcal{D}^\alpha f)(x) = K_n\,I_1(|\nabla f|)(x), \tag{5.9}$$

where K_n is given by (5.4).

Remark 5.2. The constant K_n coincides with the geometric constant appearing in the BBM limit (5.3), and it is independent of the choice of the unit vector $e \in \mathbb{S}^{n-1}$ by rotation invariance.

As a direct consequence, Theorem 5.1 shows that the inequality (5.8), proved in **[HMP25a]**, is asymptotically sharp as $\alpha \to 1^-$.

5.2 Auxiliary results

This section is devoted to the auxiliary results necessary to prove Theorem 5.1. As a first step, we provide a self-contained proof of the pointwise bound (5.8), following the arguments from **[HMP25a]**.

Theorem 5.3. *Let $0 < \alpha < 1$. Then, for every $f \in C_c^\infty(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$,*

$$(1-\alpha)\, I_\alpha(\mathcal{D}^\alpha f)(x) \le C_{n,\alpha}\, I_1(|\nabla f|)(x),$$

where $C_{n,\alpha} > 0$ remains bounded as $\alpha \to 1^-$.

Proof. By the Fundamental Theorem of Calculus for line integrals, we have

$$f(y) - f(z) = \int_0^1 \nabla f(y + t(z-y)) \cdot (z-y)\, dt.$$

Hence, by Fubini's theorem we have

$$\begin{aligned}
I_\alpha(\mathcal{D}^\alpha f)(x) =& \gamma_{n,\alpha} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)-f(z)|}{|y-z|^{n+\alpha}} dz \frac{1}{|x-y|^{n-\alpha}}\, dy \\
\le& \gamma_{n,\alpha} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 |\nabla f(y+t(z-y))|\, dt \frac{1}{|y-z|^{n+\alpha-1}} dz \frac{1}{|x-y|^{n-\alpha}}\, dy \\
=& \gamma_{n,\alpha} \int_{\mathbb{R}^n} \int_0^1 \int_{\mathbb{R}^n} \frac{|\nabla f(y+t(z-y))|}{|y-z|^{n+\alpha-1}} dz\, dt \frac{1}{|x-y|^{n-\alpha}}\, dy \\
=& \gamma_{n,\alpha} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|\nabla f(u)|}{|u-y|^{n+\alpha-1}}\, du \int_0^1 \frac{t^{n+\alpha-1}}{t^n}\, dt \right) \frac{1}{|x-y|^{n-\alpha}}\, dy \\
=& \frac{\gamma_{n,\alpha}}{\alpha} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\nabla f(u)|}{|u-y|^{n+\alpha-1}}\, du \frac{1}{|x-y|^{n-\alpha}}\, dy \\
=& \frac{\gamma_{n,\alpha}}{\alpha} \int_{\mathbb{R}^n} |\nabla f(u)| \int_{\mathbb{R}^n} \frac{1}{|u-y|^{n+\alpha-1}} \frac{1}{|x-y|^{n-\alpha}}\, dy\, du,
\end{aligned}$$

where we have made the change of variables $u = y + t(z-y)$. We now use the following identity (see **[Gra14, Exercise 2.4.9]**),

$$\int_{\mathbb{R}^n} \frac{1}{|x_1-t|^{\alpha_1}} \frac{1}{|x_2-t|^{\alpha_2}}\, dt = K_{\alpha_1,\alpha_2} |x_1-x_2|^{n-\alpha_1-\alpha_2}$$

for $0 < \alpha_1, \alpha_2 < n$ and $\alpha_1 + \alpha_2 > n$, where

$$K_{\alpha_1,\alpha_2} = \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n-\alpha_1}{2})\Gamma(\frac{n-\alpha_2}{2})\Gamma(\frac{\alpha_1+\alpha_2-n}{2})}{\Gamma(\frac{\alpha_1}{2})\Gamma(\frac{\alpha_2}{2})\Gamma(n-\frac{\alpha_1+\alpha_2}{2})}.$$

Applying this identity with $\alpha_1 = n+\alpha-1$ and $\alpha_2 = n-\alpha$, we have

$$\begin{aligned}
(1-\alpha) I_\alpha(\mathcal{D}^\alpha f)(x) \le& (1-\alpha) \frac{\gamma_{n,\alpha}}{\alpha} K_{n+\alpha-1,n-\alpha} \int_{\mathbb{R}^n} \frac{|\nabla f(u)|}{|x-u|^{n-1}}\, du \\
=& (1-\alpha) \frac{\gamma_{n,\alpha}}{\alpha\, \gamma_{n,1}} K_{n+\alpha-1,n-\alpha}\, I_1(|\nabla f|)(x).
\end{aligned}$$

Now, we compute the constant,

$$\begin{aligned}
C_{n,\alpha} :=& (1-\alpha)\frac{\gamma_{n,\alpha}}{\alpha\,\gamma_{n,1}} K_{n+\alpha-1,n-\alpha} \\
=& (1-\alpha)\frac{\pi^{\frac{n}{2}}}{\alpha}\frac{\Gamma(\frac{n-\alpha}{2})2\pi^{\frac{n}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{\alpha}{2})\Gamma(\frac{n-1}{2})}{2^{\alpha}\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})\Gamma(\frac{n-1}{2})\Gamma(\frac{n+\alpha-1}{2})\Gamma(\frac{n-\alpha}{2})\Gamma(n-\frac{2n-1}{2})} \\
=& (1-\alpha)\frac{2^{1-\alpha}\pi^{\frac{n}{2}}}{\alpha}\frac{\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{n+\alpha-1}{2})} \\
=& \frac{2^{2-\alpha}\pi^{\frac{n}{2}}}{\alpha}\frac{\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{n+\alpha-1}{2})},
\end{aligned}$$

where we have used $(1-\alpha)\Gamma(\frac{1-\alpha}{2}) = 2\Gamma(\frac{3-\alpha}{2})$. Hence, we have the limit

$$\lim_{\alpha\to 1^-} C_{n,\alpha} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \sigma(\mathbb{S}^{n-1}),$$

where $\sigma(\mathbb{S}^{n-1})$ is the surface measure of the unit sphere. □

As a first step toward the proof of the main result, we analyze the pointwise limit of the fractional gradient $\mathcal{D}^\alpha f$. The following lemma describes this behavior for smooth functions.

Lemma 5.4. *Let $f \in C_c^2(\mathbb{R}^n)$, then for every $x \in \mathbb{R}^n$ we have*

$$\lim_{\alpha\to 1^-} (1-\alpha)\mathcal{D}^\alpha f(x) = K_n\,|\nabla f(x)|.$$

The previous result is implicit in **[BBM02]**; we include a proof for the convenience of the reader. We refer to **[Moh24]** for a more general version involving general domains and mixed Sobolev seminorms $W_q^{\alpha,p}$.

Proof. Fix $x \in \mathbb{R}^n$. We split the fractional derivative into two terms, the local and nonlocal parts,

$$\begin{aligned}
(1-\alpha)\mathcal{D}^\alpha f(x) =& (1-\alpha)\int_{\mathbb{R}^n} \frac{|f(x+h)-f(x)|}{|h|^{n+\alpha}}dh \\
=& (1-\alpha)\int_{|h|<1} \frac{|f(x+h)-f(x)|}{|h|^{n+\alpha}}dh \\
&+ (1-\alpha)\int_{|h|\geq 1} \frac{|f(x+h)-f(x)|}{|h|^{n+\alpha}}dh \\
=& I_1 + I_2.
\end{aligned}$$

First, we observe that the nonlocal part does not contribute to the limit,

$$\begin{aligned}
I_2 \leq& 2\|f\|_{L^\infty}(1-\alpha)\int_{|h|\geq 1}\frac{1}{|h|^{n+\alpha}}dh \\
=& C_n\|f\|_{L^\infty}(1-\alpha)\int_1^\infty \frac{1}{r^{1+\alpha}}dr \\
=& C_n\|f\|_{L^\infty}\frac{1-\alpha}{\alpha} \longrightarrow 0,
\end{aligned}$$

when $\alpha \to 1^-$. To study the local part, we consider the Taylor expansion of f of order 2,

$$f(x+h) = f(x) + \nabla f(x) \cdot h + R(h),$$

where the remainder term satisfies $|R(h)| \leq \frac{1}{2}\|D^2 f\|_{L^\infty} |h|^2$ for $|h| < 1$. Using the elementary inequality $\big||a+b| - |a|\big| \leq |b|$ for $a, b \in \mathbb{R}$, with $a = \nabla f(x) \cdot h$ and $b = R(h)$, we obtain

$$\big||f(x+h) - f(x)| - |\nabla f(x) \cdot h|\big| \leq |R(h)|, \qquad |h| < 1.$$

Therefore,

$$|I_1 - I_{11}| \leq I_{12}, \tag{5.10}$$

where

$$I_{11} := (1-\alpha) \int_{|h|<1} \frac{|\nabla f(x) \cdot h|}{|h|^{n+\alpha}}\, dh,$$

and,

$$I_{12} := (1-\alpha) \int_{|h|<1} \frac{|R(h)|}{|h|^{n+\alpha}}\, dh.$$

The remainder term satisfies

$$\begin{aligned} I_{12} &\leq C_f (1-\alpha) \int_{|h|<1} \frac{|h|^2}{|h|^{n+\alpha}}\, dh \\ &= C_f C_n (1-\alpha) \int_0^1 r^{1-\alpha}\, dr \\ &= C_f C_n \frac{1-\alpha}{2-\alpha} \longrightarrow 0, \end{aligned}$$

when $\alpha \to 1^-$. Next, we compute I_{11}. If $\nabla f(x) = 0$, then $I_{11} = 0$ and (5.10) gives $0 \leq I_1 \leq I_{12} \longrightarrow 0$, and then $\lim_{\alpha \to 1^-} (1-\alpha)\mathcal{D}^\alpha f(x) = 0 = K_n |\nabla f(x)|$. Assume now that $\nabla f(x) \neq 0$. Using polar coordinates $h = r\omega$, we get

$$\begin{aligned} I_{11} &= (1-\alpha) \int_0^1 \int_{\mathbb{S}^{n-1}} \frac{|\nabla f(x) \cdot (r\omega)|}{r^{n+\alpha}}\, r^{n-1}\, d\sigma(\omega)\, dr \\ &= (1-\alpha) \left(\int_{\mathbb{S}^{n-1}} |\nabla f(x) \cdot \omega|\, d\sigma(\omega) \right) \int_0^1 r^{-\alpha}\, dr \\ &= (1-\alpha) \left(\int_{\mathbb{S}^{n-1}} |\nabla f(x) \cdot \omega|\, d\sigma(\omega) \right) \frac{1}{1-\alpha} \\ &= \int_{\mathbb{S}^{n-1}} |\nabla f(x) \cdot \omega|\, d\sigma(\omega). \end{aligned}$$

By homogeneity and rotational invariance,

$$\int_{\mathbb{S}^{n-1}} |\nabla f(x) \cdot \omega|\, d\sigma(\omega) = |\nabla f(x)| \int_{\mathbb{S}^{n-1}} \left| \frac{\nabla f(x)}{|\nabla f(x)|} \cdot \omega \right| d\sigma(\omega) = K_n |\nabla f(x)|.$$

Therefore $I_{11} = K_n |\nabla f(x)|$ and, since $I_{12} \to 0$, (5.10) yields $\lim_{\alpha \to 1^-} I_1 = K_n |\nabla f(x)|$. Together with $I_2 \to 0$ we conclude that

$$\lim_{\alpha \to 1^-} (1-\alpha)\mathcal{D}^\alpha f(x) = \lim_{\alpha \to 1^-} (I_1 + I_2) = K_n |\nabla f(x)|.$$

This concludes the proof. □

Remark 5.5. As demonstrated in the proof, the nonlocal term vanishes in the limit. Actually, we have established the following result:

$$\lim_{\alpha\to 1^-}(1-\alpha)\int_{B(x,r)}\frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}dy = K_n\,|\nabla f(x)|,$$

for every $x\in\mathbb{R}^n$ and every $0<r\le\infty$. In particular, if Ω is a domain and $\tau\in(0,1)$ we have proved

$$\lim_{\alpha\to 1^-}(1-\alpha)\int_{B(x,\tau\,\mathrm{dist}(x,\partial\Omega))}\frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}dy = K_n\,|\nabla f(x)|, \tag{5.11}$$

for every $x\in\Omega$.

To justify the exchange of limits and integrals later in the proof of Theorem 5.1, we need a uniform pointwise bound for $(1-\alpha)\mathcal{D}^\alpha f$ that holds uniformly for α close to 1.

Lemma 5.6. *Let $f\in C_c^2(\mathbb{R}^n)$, then for $\alpha\in(\frac{1}{2},1)$ and for each $x\in\mathbb{R}^n$ we have*

$$(1-\alpha)\mathcal{D}^\alpha f(x)\le C_n\left(\|f\|_{L^\infty}+\|\nabla f\|_{L^\infty}\right).$$

Proof. We again split the fractional derivative into two terms: the local and nonlocal parts. We have

$$\begin{aligned}(1-\alpha)\mathcal{D}^\alpha f(x) =&(1-\alpha)\int_{|x-y|<1}\frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}dy\\ &+(1-\alpha)\int_{|x-y|\ge 1}\frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}dy\\ =&I_1+I_2.\end{aligned}$$

To estimate the local term, by the mean value theorem, we have

$$\begin{aligned}I_1\le&C\|\nabla f\|_{L^\infty}(1-\alpha)\int_{|x-y|<1}\frac{|x-y|}{|x-y|^{n+\alpha}}dy\\ =&C_n\|\nabla f\|_{L^\infty}(1-\alpha)\int_0^1 r^{-\alpha}dr\\ =&C_n\|\nabla f\|_{L^\infty}.\end{aligned}$$

On the other hand, we use the fact that f is bounded,

$$\begin{aligned}I_2\le&2\|f\|_{L^\infty}(1-\alpha)\int_{|x-y|\ge 1}\frac{1}{|x-y|^{n+\alpha}}dy\\ =&C_n\|f\|_{L^\infty}(1-\alpha)\int_1^\infty r^{-1-\alpha}dr\\ =&C_n\|f\|_{L^\infty}\frac{1-\alpha}{\alpha}\\ \le&C_n\|f\|_{L^\infty},\end{aligned}$$

where in the last inequality we use $\alpha\in(\frac{1}{2},1)$. □

Remark 5.7. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain and fix $\tau \in (0,1)$. Combining (5.11) with the uniform bound from Lemma 5.6, one may apply the dominated convergence theorem to obtain the well-known BBM limit in Ω with the truncated seminorm

$$\lim_{\alpha\to 1^-}(1-\alpha)\int_\Omega \int_{B(x,\tau\ \mathrm{dist}(x,\partial\Omega))} \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dy\,d\mu(x) = K_n \int_\Omega |\nabla f(x)|\,d\mu(x),$$

where μ is a locally finite measure. In particular, this recovers the main result of **[DD22]** for $p = 1$ and $f \in C_c^2(\mathbb{R}^n)$.

5.3 Proof of Theorem 5.1

We are now ready to prove the main result of this chapter.

Proof of Theorem 5.1. Fix $x \in \mathbb{R}^n$ and set $f_x(u) := f(u+x)$. Since both $\mathcal{D}^\alpha$ and the Riesz potential I_α are translation invariant, we have, for every $\alpha \in (0,1)$,

$$(1-\alpha)\, I_\alpha(\mathcal{D}^\alpha f)(x) = (1-\alpha)\, I_\alpha(\mathcal{D}^\alpha f_x)(0).$$

Moreover, $|\nabla f_x(u)| = |\nabla f(u+x)|$, and therefore

$$I_1(|\nabla f|)(x) = I_1(|\nabla f_x|)(0).$$

Hence, it is enough to prove (5.9) in the case $x = 0$, applied to the translated function f_x. In what follows, we assume $x = 0$ and write f in place of f_x for simplicity.

For each $\alpha \in (0,1)$, define

$$F_\alpha(y) = \gamma_{n,\alpha}(1-\alpha)\frac{\mathcal{D}^\alpha f(y)}{|y|^{n-\alpha}},$$

and

$$F(y) = K_n\gamma_{n,1}\frac{|\nabla f(y)|}{|y|^{n-1}},$$

for $y \in \mathbb{R}^n \setminus \{0\}$. By Lemma 5.4, together with the continuity of $\gamma_{n,\alpha}$ as a function of α, we have

$$\lim_{\alpha\to 1^-} F_\alpha(y) = F(y),$$

for every $y \in \mathbb{R}^n \setminus \{0\}$. Thus, it remains to show that

$$\lim_{\alpha\to 1^-}\int_{\mathbb{R}^n} F_\alpha(y)\,dy = \int_{\mathbb{R}^n} F(y)\,dy. \tag{5.12}$$

To justify the interchange of limit and integral, we shall apply the dominated convergence theorem. Let $R_0 > 0$ be such that $\mathrm{supp}(f) \subset B(0,R_0)$, and decompose

$$\mathbb{R}^n = A_1 \cup A_2 \cup A_3,$$

where

$$A_1 = B(0,1), \qquad A_2 = \mathbb{R}^n \setminus B(0,2R_0+1), \qquad A_3 = B(0,2R_0+1)\setminus B(0,1).$$

We begin with $y \in A_1$. If $\alpha \in (\frac{1}{2}, 1)$, then Lemma 5.6 gives

$$\begin{aligned} F_\alpha(y) &= \gamma_{n,\alpha}(1-\alpha)\frac{\mathcal{D}^\alpha f(y)}{|y|^{n-\alpha}} \\ &\le C_n \frac{\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}}{|y|^{n-\alpha}} \\ &\le C_{n,f}\frac{1}{|y|^{n-\frac{1}{2}}} \\ &=: h(y). \end{aligned}$$

Moreover, $h \in L^1(A_1)$, since

$$\int_{A_1} h(y)\,dy = \int_{B(0,1)} \frac{1}{|y|^{n-\frac{1}{2}}}\,dy = C_n \int_0^1 r^{-\frac{1}{2}}\,dr < \infty.$$

Next, let $y \in A_2$. Then $f(y) = 0$, and therefore

$$\begin{aligned} \mathcal{D}^\alpha f(y) &= \int_{\mathbb{R}^n} \frac{|f(y)-f(z)|}{|y-z|^{n+\alpha}}\,dz \\ &= \int_{\mathrm{supp}(f)} \frac{|f(z)|}{|y-z|^{n+\alpha}}\,dz \\ &\le \int_{\mathrm{supp}(f)} |f(z)|\frac{2^{n+\alpha}}{|y|^{n+\alpha}}\,dz \\ &= 2^{n+\alpha}\|f\|_{L^1}\frac{1}{|y|^{n+\alpha}}, \end{aligned}$$

because $z \in \mathrm{supp}(f) \subset B(0,R_0)$ and $|y| \ge 2R_0 + 1$, so

$$|y-z| \ge |y| - |z| \ge |y| - R_0 \ge \frac{|y|}{2}.$$

Hence, for $y \in A_2$

$$\begin{aligned} F_\alpha(y) &= \gamma_{n,\alpha}(1-\alpha)\frac{\mathcal{D}^\alpha f(y)}{|y|^{n-\alpha}} \\ &\le \gamma_{n,\alpha}(1-\alpha)2^{n+\alpha}\|f\|_{L^1}\frac{1}{|y|^{n+\alpha}|y|^{n-\alpha}} \\ &\le C_n\|f\|_{L^1}\frac{1}{|y|^{2n}} \\ &=: g(y), \end{aligned}$$

and $g \in L^1(A_2)$, since

$$\int_{A_2} g(y)\,dy = C_{n,f}\int_{\mathbb{R}^n\setminus B(0,2R_0+1)} \frac{1}{|y|^{2n}}\,dy = C_{n,f}\int_{2R_0+1}^\infty r^{-n-1}\,dr < \infty.$$

Finally, let $y \in A_3$. Since $|y| \ge 1$, Lemma 5.6 yields

$$F_\alpha(y) = \gamma_{n,\alpha}(1-\alpha)\frac{\mathcal{D}^\alpha f(y)}{|y|^{n-\alpha}} \le C_{n,f},$$

for every $\alpha \in (\frac{1}{2}, 1)$. Since $|A_3| < \infty$, this gives an integrable dominating function on A_3.

Combining the estimates on A_1, A_2, and A_3, we obtain an L^1-dominating function on $\mathbb{R}^n$. Therefore, the dominated convergence theorem yields (5.12) which completes the proof. □

5.4 Extension to the Sobolev space $W^{1,1}(\mathbb{R}^n)$

In this section, we extend Theorem 5.1 to the Sobolev space $W^{1,1}(\mathbb{R}^n)$ by a density argument. We begin with a standard result showing that local convergence in measure implies almost everywhere convergence along a subsequence.

Proposition 5.8. *Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}^n$ and let f be measurable. Assume that $f_n \to f$ in measure on every compact set $K \subset \mathbb{R}^n$. Then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \to f(x)$ for almost every $x \in \mathbb{R}^n$.*

Proof. Set $K_m := \overline{B(0,m)}$ for $m \in \mathbb{N}$. Since $f_n \to f$ in measure on K_m and $|K_m| < \infty$, there exists a subsequence $\{f_k^{(m)}\}_k$ of $\{f_n\}$ such that $f_k^{(m)}(x) \to f(x)$ for almost every $x \in K_m$. Choosing these subsequences inductively, we may assume that

$$\{f_k^{(1)}\}_k \supset \{f_k^{(2)}\}_k \supset \cdots .$$

Define the diagonal subsequence by $g_k := f_k^{(k)}$. Fix $m \in \mathbb{N}$. Then $\{g_k\}_{k \geq m}$ is a subsequence of $\{f_k^{(m)}\}_k$, hence $g_k(x) \to f(x)$ for almost every $x \in K_m$. Let $N_m \subset K_m$ be the corresponding null set, and set $N := \bigcup_{m=1}^{\infty} N_m$. Then $|N| = 0$, and since $\mathbb{R}^n = \bigcup_{m=1}^{\infty} K_m$, we conclude that $g_k(x) \to f(x)$ for every $x \in \mathbb{R}^n \setminus N$. □

Now, we prove the following estimate for $p = 1$ and the global Gagliardo seminorm.

Proposition 5.9. *Let $f \in W^{1,1}(\mathbb{R}^n)$ and let $0 < \alpha < 1$. Then*

$$\alpha(1-\alpha) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+\alpha}} \, dy \, dx \leq C_n \|f\|_{W^{1,1}(\mathbb{R}^n)}.$$

This proposition is a consequence of the Gagliardo–Nirenberg interpolation inequality proved in **[BM18]** (see also **[HLYY25, (1.10)]**). For completeness, we include a proof.

Proof. By the change of variables $y = x + h$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+\alpha}} \, dy \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|}{|h|^{n+\alpha}} \, dx \, dh =: I_1 + I_2,$$

where in I_1 we integrate over $|h| < 1$, while in I_2 we integrate over $|h| \geq 1$. For I_1, we use the following difference quotients estimate,

$$\int_{\mathbb{R}^n} |f(x+h) - f(x)| \, dx \leq |h| \int_{\mathbb{R}^n} |\nabla f(x)| \, dx,$$

for every $h \in \mathbb{R}^n$ (see for instance **[Bre11, Proposition 9.3]**). Indeed, this is immediate for $f \in C_c^{\infty}(\mathbb{R}^n)$ from

$$f(x+h) - f(x) = \int_0^1 \nabla f(x+th) \cdot h \, dt,$$

and the general case follows by density in $W^{1,1}(\mathbb{R}^n)$. Hence

$$I_1 \leq \int_{|h|<1} \frac{1}{|h|^{n+\alpha-1}} \left(\int_{\mathbb{R}^n} |\nabla f(x)| \, dx \right) dh \leq \frac{C_n}{1-\alpha} \int_{\mathbb{R}^n} |\nabla f(x)| \, dx.$$

For I_2, the triangle inequality gives

$$I_2 \le 2 \int_{|h|\ge 1} \frac{dh}{|h|^{n+\alpha}} \int_{\mathbb{R}^n} |f(x)|\,dx \le \frac{C_n}{\alpha} \int_{\mathbb{R}^n} |f(x)|\,dx.$$

Combining both estimates, we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dy\,dx \le \frac{C_n}{1-\alpha}\int_{\mathbb{R}^n} |\nabla f(x)|\,dx + \frac{C_n}{\alpha}\int_{\mathbb{R}^n} |f(x)|\,dx.$$

We conclude the proof by multiplying both sides by $\alpha(1-\alpha)$ and using that $\alpha < 1$ and $1-\alpha < 1$. □

With these preliminary results at hand, we can now extend the main theorem to Sobolev functions.

Theorem 5.10. *Let $f \in W^{1,1}(\mathbb{R}^n)$ with $n \ge 2$. Then, for every sequence $\{\alpha_k\}_k \subset (0,1)$ with $\alpha_k \to 1^-$, there exists a subsequence $\{\alpha_{k_j}\}_j$ such that*

$$\lim_{j\to\infty} (1-\alpha_{k_j}) I_{\alpha_{k_j}}(\mathcal{D}^{\alpha_{k_j}} f)(x) = K_n\, I_1(|\nabla f|)(x),$$

for almost every $x \in \mathbb{R}^n$.

Proof. Let $f \in W^{1,1}(\mathbb{R}^n)$. By the density of smooth compactly supported functions, there exists a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset C_c^\infty(\mathbb{R}^n)$ such that $\varphi_k \to f$ in $W^{1,1}(\mathbb{R}^n)$ (see for instance **[AF03; Bre11]**). We first show that

$$(1-\alpha) I_\alpha(\mathcal{D}^\alpha f) \to K_n I_1(|\nabla f|)$$

in measure on every compact set $E \subset \mathbb{R}^n$ as $\alpha \to 1^-$.

For every $k \in \mathbb{N}$ and every $\alpha \in (0,1)$, we use the triangle inequality to write

$$\begin{aligned}
|(1-\alpha) I_\alpha(\mathcal{D}^\alpha f)(x) - K_n I_1(|\nabla f|)(x)| &\le (1-\alpha)\,|I_\alpha(\mathcal{D}^\alpha f)(x) - I_\alpha(\mathcal{D}^\alpha \varphi_k)(x)| \\
&\quad + |(1-\alpha) I_\alpha(\mathcal{D}^\alpha \varphi_k)(x) - K_n I_1(|\nabla \varphi_k|)(x)| \\
&\quad + K_n\,|I_1(|\nabla \varphi_k|)(x) - I_1(|\nabla f|)(x)| \\
&=: A_{1,\alpha,k}(x) + A_{2,\alpha,k}(x) + A_{3,k}(x).
\end{aligned}$$

We begin by bounding the first term. Combining the weak-type estimate for the Riesz potential with Proposition 5.9 for $\alpha \in (\frac{1}{2}, 1)$, we obtain

$$\begin{aligned}
\|A_{1,\alpha,k}\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} &\le (1-\alpha)\|I_\alpha(\mathcal{D}^\alpha(f-\varphi_k))\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} \\
&\le C_n(1-\alpha)\int_{\mathbb{R}^n} \mathcal{D}^\alpha(f-\varphi_k)(x)\,dx \\
&\le C_n \|f-\varphi_k\|_{W^{1,1}(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, for the third term,

$$\begin{aligned}
\|A_{3,k}\|_{L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)} &\le K_n \|I_1(|\nabla\varphi_k - \nabla f|)\|_{L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)} \\
&\le C_n \|\nabla\varphi_k - \nabla f\|_{L^1(\mathbb{R}^n)} \\
&= C_n \|\nabla(\varphi_k - f)\|_{L^1(\mathbb{R}^n)} \\
&\le C_n \|f-\varphi_k\|_{W^{1,1}(\mathbb{R}^n)}.
\end{aligned}$$

Given any $\varepsilon > 0$ and $\eta > 0$, since $\varphi_k \to f$ in $W^{1,1}(\mathbb{R}^n)$, we can choose k sufficiently large such that

$$\frac{C_n\|f-\varphi_k\|_{W^{1,1}(\mathbb{R}^n)}}{\varepsilon} < \min\left\{1, \frac{\eta}{3}, \left(\frac{\eta}{3}\right)^{\frac{n-1}{n}}\right\}.$$

Fix a compact set $E \subset \mathbb{R}^n$. Since $\frac{n}{n-\alpha} \geq 1$ for every $\alpha \in (0,1)$, applying Chebyshev's inequality to $A_{1,\alpha,k}$ we have

$$\begin{aligned}
\left|\left\{x \in E : \ A_{1,\alpha,k}(x) > \varepsilon\right\}\right| &\leq \left|\left\{x \in \mathbb{R}^n : \ A_{1,\alpha,k}(x) > \varepsilon\right\}\right| \\
&\leq \left(\frac{\|A_{1,\alpha,k}\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)}}{\varepsilon}\right)^{\frac{n}{n-\alpha}} \\
&\leq \left(\frac{C_n\|f-\varphi_k\|_{W^{1,1}(\mathbb{R}^n)}}{\varepsilon}\right)^{\frac{n}{n-\alpha}} \\
&\leq \frac{C_n\|f-\varphi_k\|_{W^{1,1}(\mathbb{R}^n)}}{\varepsilon} \\
&< \frac{\eta}{3},
\end{aligned}$$

where we have used the fact that $\frac{C_n\|f-\varphi_k\|_{W^{1,1}(\mathbb{R}^n)}}{\varepsilon}$ is strictly less than 1 and the exponent is larger than 1. Applying Chebyshev's inequality to $A_{3,k}$, we also obtain

$$\begin{aligned}
\left|\left\{x \in E : \ A_{3,k}(x) > \varepsilon\right\}\right| &\leq \left(\frac{\|A_{3,k}\|_{L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)}}{\varepsilon}\right)^{\frac{n}{n-1}} \\
&\leq \left(\frac{C_n\|f-\varphi_k\|_{W^{1,1}(\mathbb{R}^n)}}{\varepsilon}\right)^{\frac{n}{n-1}} \\
&< \frac{\eta}{3}.
\end{aligned}$$

For this fixed choice of k, since $\varphi_k \in C_c^\infty(\mathbb{R}^n)$, we may apply Theorem 5.1 to conclude that $A_{2,\alpha,k}(x) \to 0$ for every $x \in \mathbb{R}^n$ as $\alpha \to 1^-$. Since $|E| < \infty$, pointwise convergence implies convergence in measure on E. Thus, there exists $\alpha_0 \in (\frac{1}{2}, 1)$ (depending on ε and η) such that for every $\alpha \in (\alpha_0, 1)$,

$$\left|\left\{x \in E : \ A_{2,\alpha,k}(x) > \varepsilon\right\}\right| < \frac{\eta}{3}.$$

Combining the estimates for $A_{1,\alpha,k}$, $A_{2,\alpha,k}$, and $A_{3,k}$ yields

$$|\{x \in E : \ |(1-\alpha)I_\alpha(\mathcal{D}^\alpha f)(x) - K_n I_1(|\nabla f|)(x)| > 3\varepsilon\}| < \eta,$$

for every $\alpha \in (\alpha_0, 1)$. Since $\varepsilon, \eta > 0$ were arbitrary, this proves that $(1-\alpha)I_\alpha(\mathcal{D}^\alpha f) \to K_n I_1(|\nabla f|)$ in measure on every compact set $E \subset \mathbb{R}^n$ as $\alpha \to 1^-$.

Finally, let $\{\alpha_k\}_k \subset (0,1)$ be any sequence such that $\alpha_k \to 1^-$. By the previous step, the sequence $\{(1-\alpha_k)I_{\alpha_k}(\mathcal{D}^{\alpha_k} f)\}_k$ converges to $K_n I_1(|\nabla f|)$ in measure on every compact subset of $\mathbb{R}^n$. Therefore, by applying Proposition 5.8 there exists a subsequence $\{\alpha_{k_j}\}_j$ such that

$$(1-\alpha_{k_j})I_{\alpha_{k_j}}(\mathcal{D}^{\alpha_{k_j}} f)(x) \to K_n I_1(|\nabla f|)(x),$$

for almost every $x \in \mathbb{R}^n$. This completes the proof. □

5.5 Further extensions

In this final section, we record a natural L^p-variant of the nonlinear fractional differential operator. For $0 < \alpha < 1$ and $p \geq 1$, we define

$$\mathcal{D}_p^\alpha f(x) := \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha p}}\, dy \right)^{\frac{1}{p}}. \tag{5.13}$$

This operator (in particular for $p = 2$) appears, for instance, in **[NP09]**. Moreover, it behaves like a differential operator, in the sense that it satisfies the following Leibniz-type inequality.

Proposition 5.11. *Let $0 < \alpha < 1$ and $1 \leq p < \infty$. Then,*

$$\left\| \mathcal{D}_p^\alpha (fg) \right\|_{L^p(\mathbb{R}^n)} \leq \left\| f \mathcal{D}_p^\alpha g \right\|_{L^p(\mathbb{R}^n)} + \left\| g \mathcal{D}_p^\alpha f \right\|_{L^p(\mathbb{R}^n)}.$$

Proof. Using the identity

$$f(x)g(x) - f(y)g(y) = f(x)\big(g(x) - g(y)\big) + g(y)\big(f(x) - f(y)\big),$$

and applying Minkowski's inequality, we obtain

$$\begin{aligned} \left\| \mathcal{D}_p^\alpha (fg) \right\|_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)g(x) - f(y)g(y)|^p}{|x-y|^{n+\alpha p}}\, dy\, dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)|^p |g(x) - g(y)|^p}{|x-y|^{n+\alpha p}}\, dy\, dx \right)^{\frac{1}{p}} \\ &\quad + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(y)|^p |f(x) - f(y)|^p}{|x-y|^{n+\alpha p}}\, dy\, dx \right)^{\frac{1}{p}}. \end{aligned}$$

The first term is exactly $\left\| f \mathcal{D}_p^\alpha g \right\|_{L^p(\mathbb{R}^n)}$. For the second one, by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(y)|^p |f(x) - f(y)|^p}{|x-y|^{n+\alpha p}}\, dy\, dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(x)|^p |f(x) - f(y)|^p}{|x-y|^{n+\alpha p}}\, dy\, dx \\ &= \left\| g \mathcal{D}_p^\alpha f \right\|_{L^p(\mathbb{R}^n)}^p. \end{aligned}$$

Combining the two estimates gives the desired inequality. □

We next state the corresponding pointwise BBM limit for the Riesz potential of $\mathcal{D}_p^\alpha$.

Theorem 5.12. *Let $p \geq 1$ and let $f \in C_c^\infty(\mathbb{R}^n)$ with $n \geq 2$. Then for every $x \in \mathbb{R}^n$,*

$$\lim_{\alpha \to 1^-} (1-\alpha)^{\frac{1}{p}}\, I_\alpha(\mathcal{D}_p^\alpha f)(x) = K_{n,p}\, I_1(|\nabla f|)(x),$$

where

$$K_{n,p} := \left(\frac{1}{p} \int_{\mathbb{S}^{n-1}} |\omega \cdot e|^p\, d\sigma(\omega) \right)^{\frac{1}{p}}.$$

Sketch of the proof. The proof follows the strategy of Theorem 5.1. By translation invariance, it suffices to consider $x = 0$. Setting

$$F_{\alpha,p}(y) := \gamma_{n,\alpha}\,(1-\alpha)^{\frac{1}{p}}\,\frac{\mathcal{D}_p^\alpha f(y)}{|y|^{n-\alpha}}, \qquad F(y) := K_{n,p}\,\gamma_{n,1}\,\frac{|\nabla f(y)|}{|y|^{n-1}},$$

it is enough to justify that $\lim_{\alpha\to 1^-}\int_{\mathbb{R}^n} F_{\alpha,p} = \int_{\mathbb{R}^n} F$. The pointwise convergence $F_{\alpha,p}(y) \to F(y)$ is provided by **[Moh24, Lemma 13]** (in place of Lemma 5.4) together with the continuity of $\gamma_{n,\alpha}$ in α. The required domination to apply the dominated convergence theorem is obtained by the same splitting estimates as in the proof of Theorem 5.1, adapted to $\mathcal{D}_p^\alpha f$. □

This shows that the pointwise BBM phenomenon established in this chapter is not restricted to the L^1-based nonlinear fractional differential operator, but extends naturally to the corresponding L^p-scale.

Chapter 6

On BMO^β, sharp maximal functions, and commutators of fractional integrals

In this chapter, we present the results of the two following papers.

[CC25] Chen, You-Wei Benson and Claros, A. *β-dimensional sharp maximal function and applications*, arXiv preprint arXiv:2407.04456 (2025).

[CC26] Chen, You-Wei Benson and Claros, A. *Commutators of Fractional Integrals with* BMO^β *Functions*, arXiv preprint arXiv:2602.09742 (2026).

This chapter deals with a topic that is rather different from those of the preceding chapters. Here we study harmonic analysis in a setting where Hausdorff content replaces Lebesgue measure. My interest in this topic began at the conference GEOFI 2023, held in Prague, where Daniel Spector gave a talk on functions of bounded β-dimensional mean oscillation. Later that same year, during the Spring School on Analysis 2023 in Paseky, Daniel Spector introduced me to You-Wei Benson Chen, who was at that time his PhD student. During that school, Daniel Spector proposed to us the problem that eventually led to **[CC25]**. Later, during a visit to the National Taiwan Normal University, David Cruz-Uribe suggested to You-Wei Benson Chen the problem that eventually became **[CC26]**.

6.1 Introduction

In the study of Sobolev spaces, Lebesgue measure is often too coarse to capture fine properties of functions, such as pointwise behavior and trace theory. For these questions, capacity provides the right notion of size, and this point of view is standard in nonlinear potential theory and Sobolev theory (see for instance **[AH96; EG15; Maz85; Zie89]**).

Let $0 < \alpha < n$ and $1 \le p < \frac{n}{\alpha}$. We define the Riesz (α, p)-capacity of an arbitrary set $E \subset \mathbb{R}^n$ by

$$\mathrm{cap}_{\alpha,p}(E) := \inf \left\{ \|f\|^p_{L^p(\mathbb{R}^n)} : f \in \mathcal{S}(\mathbb{R}^n),\ f \ge 0,\ I_\alpha f \ge 1 \text{ on } E \right\},$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class,

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \varphi \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\gamma \partial^\eta \varphi(x)| < \infty \text{ for all multi-indices } \gamma, \eta \right\}.$$

The resulting set function is a Choquet capacity (see Section 6.2 for a precise definition). In general, capacities are not additive set functions.

The capacitary point of view is natural in the theory of Sobolev functions. Indeed, in the subcritical case $1 < p < n$, every function in $W^{1,p}(\mathbb{R}^n)$ admits a quasicontinuous representative, unique up to sets of $\mathrm{cap}_{1,p}$-capacity zero. In particular, the relevant exceptional sets for $W^{1,p}(\mathbb{R}^n)$ are detected by the corresponding capacity, whereas Lebesgue measure is in general too coarse to capture them (see for instance **[AH96; Maz85; Zie89]**). For the borderline and supercritical regimes $p \geq n$, however, it is more natural to work with Sobolev/Bessel capacities; we refer to **[AH96; Maz85]** and do not pursue this here.

A classical theorem of Adams **[Ada71]** shows that if $0 < \alpha < n$ and $1 < p < q < \infty$, then

$$\left(\int_{\mathbb{R}^n} |I_\alpha f|^q \, d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f|^p \, dx \right)^{\frac{1}{p}},$$

for all $f \in L^p(\mathbb{R}^n)$ if and only if

$$\mu(B(x,r)) \lesssim r^\beta,$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where

$$\beta = \frac{q}{p}(n - \alpha p).$$

Equivalently, μ is in the Morrey space of dimension β (see (6.9) below for the precise definition). We also refer to **[Ada72; Ada75; AH96]** for related trace inequalities in potential theory, and to **[Ada09]** for historical overview. At the endpoint $p = q = 1$, Adams proved that

$$\int_{\mathbb{R}^n} |I_\alpha f| \, d\mu \leq C \int_{\mathbb{R}^n} |f| \, dx,$$

for all $f \in L^1(\mathbb{R}^n)$ if and only if

$$\mu(K) \leq C \, \mathcal{H}^{n-\alpha}_\infty(K),$$

for every compact set $K \subset \mathbb{R}^n$ (see **[Ada75; AH96]**). In particular, at the endpoint the trace inequality is characterized in terms of Hausdorff content. Given $0 < \beta \leq n$, the spherical β-dimensional Hausdorff content of a set $E \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^\beta_\infty(E) := \inf \left\{ \sum_{j=1}^\infty \omega_\beta r_j^\beta : E \subset \bigcup_{j=1}^\infty B(x_j, r_j) \right\},$$

where $\omega_\beta = \pi^{\beta/2}/\Gamma(\frac{\beta}{2} + 1)$. For background on Hausdorff measures we refer to the monograph **[Mat95]**.

The set function $\mathcal{H}^\beta_\infty$ is an outer measure, but in general not a measure. More precisely, it is a capacity in the sense of Meyers, although not, in general, a capacity in the sense of Choquet. Another fundamental point is that $\mathcal{H}^\beta_\infty$ and the β-dimensional Hausdorff measure $\mathcal{H}^\beta$ have exactly the same

null sets: a set $E \subset \mathbb{R}^n$ satisfies $\mathcal{H}^\beta_\infty(E) = 0$ if and only if $\mathcal{H}^\beta(E) = 0$ (see **[AH96, Proposition 5.1.5]**). Thus, although $\mathcal{H}^\beta_\infty$ is more flexible from the point of view of amount of measurable sets, it detects precisely the same negligible sets at dimension β. In relation with the Riesz capacity, since $\mathrm{cap}_{\alpha,p}(B(x,r)) = \mathrm{cap}_{\alpha,p}(B(0,1))\, r^{n-\alpha p}$ (see **[AH96, Proposition 5.1.2]**), a standard covering argument shows that

$$\mathrm{cap}_{\alpha,p}(E) \lesssim \mathcal{H}^{n-\alpha p}_\infty(E),$$

for every set $E \subset \mathbb{R}^n$. In addition, if $\mathrm{cap}_{\alpha,p}(E) = 0$, then $\mathcal{H}^s(E) = \mathcal{H}^s_\infty(E) = 0$ for every $s > n - \alpha p$ (see **[EG15, Theorem 4.17]**). Thus, in the Sobolev case $\alpha = 1$, this shows that functions in $W^{1,p}(\mathbb{R}^n)$ are defined up to sets of $\mathcal{H}^{n-p+\varepsilon}$-measure zero for every $\varepsilon > 0$.

At this point we return to the endpoint behavior of Sobolev embeddings. In the first-order critical regime $p = n$ we have

$$W^{1,n}(Q) \hookrightarrow \mathrm{BMO}(Q).$$

Moreover, critical Sobolev spaces also admit lower-dimensional traces. If H is an $(n-1)$-dimensional affine hyperplane parallel to a coordinate hyperplane and intersecting Q, then trace theory (we refer to the original reference **[Gag57]** and the modern monograph **[Leo23, Section 9.3]**) gives

$$W^{1,n}(Q) \hookrightarrow W^{1-\frac{1}{n},n}(Q \cap H) \hookrightarrow \mathrm{BMO}(Q \cap H),$$

where $\mathrm{BMO}(Q \cap H)$ denotes the usual BMO space on the hyperplane H, with respect to the induced $(n-1)$-dimensional Lebesgue measure. Iterating this argument, we obtain

$$W^{1,n}(Q) \hookrightarrow W^{1-\frac{1}{n},n}(Q \cap H) \hookrightarrow \cdots \hookrightarrow W^{1-\frac{n-1}{n},n}(Q \cap \ell) \hookrightarrow \mathrm{BMO}(Q \cap \ell),$$

where ℓ is a one-dimensional affine subspace parallel to one of the edges of Q. At the j-th step one obtains a trace in the fractional critical space $W^{1-\frac{j}{n},n}$ on an affine subspace of dimension $n-j$, and the identity

$$\left(1 - \frac{j}{n}\right) n = n - j$$

shows that one remains exactly in the critical regime for these fractional Sobolev spaces. In this sense, functions in $W^{1,n}(Q)$ restrict to BMO on affine subspaces of every dimension from n down to 1.

Motivated by these considerations, Chen and Spector introduced in **[CS24]** the spaces BMO^β, where the oscillation is measured with respect to the β-dimensional Hausdorff content rather than with respect to Lebesgue measure. Let Q_0 be either a fixed cube in $\mathbb{R}^n$ or the whole space $\mathbb{R}^n$. For $0 < \beta \le n$, we define

$$\|f\|_{\mathrm{BMO}^\beta(Q_0)} := \sup_{Q \subset Q_0} \inf_{c \in \mathbb{R}} \frac{1}{\ell(Q)^\beta} \int_Q |f - c|\, d\mathcal{H}^\beta_\infty,$$

where the integral is understood in the sense of Choquet (see Section 6.2). When $\beta = n$, this definition coincides with the classical space $\mathrm{BMO}(Q_0)$, although the seminorm above is meaningful even for functions that are not Lebesgue measurable. For $\beta < n$, the resulting space is strictly smaller. In fact, the scale is monotone with respect to the dimensional parameter: if $0 < \alpha \le \beta \le n$, then

$$\mathrm{BMO}^\alpha(Q_0) \subset \mathrm{BMO}^\beta(Q_0).$$

Thus, smaller values of the parameter correspond to stronger oscillation control. Moreover, these spaces are well adapted to restrictions onto lower-dimensional affine subspaces: if $f \in \mathrm{BMO}^\beta(\mathbb{R}^n)$, then the restriction of f belongs to the classical space $\mathrm{BMO}(\mathbb{R}^k)$ on every k-dimensional affine subspace such that $k \geq \beta$. In this sense, BMO^β provides a space scale compatible with the lower-dimensional trace phenomena discussed above. In fact, one can prove that

$$W^{1,n}(Q) \hookrightarrow \mathrm{BMO}^\beta(Q),$$

for every $\beta \in (0, n]$, and even a stronger result holds. Specifically, it was proved in **[CMS24]** that if f is a function on $\mathbb{R}^n$ with weak gradient in $L^{n,\infty}(\mathbb{R}^n)$, then

$$\|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \leq C_{n,\beta} \|\nabla f\|_{L^{n,\infty}(\mathbb{R}^n)},$$

for every $\beta \in (0, n]$. Moreover, that paper shows that the spaces BMO^β are not only natural from the point of view of critical Sobolev theory, but also useful in the study of regularity theory of elliptic PDEs.

The motivation to define function spaces measured with respect to the Hausdorff content $\mathcal{H}^\beta_\infty$ arises naturally from the study of sharp forms of the Sobolev embedding in the critical exponent (see, e.g., **[Jud61; Ada72; Cia08; FM11; MS21]**). In this borderline regime, one often encounters not only the classical exponential integrability phenomenon, but also refined inequalities that capture concentration effects along lower-dimensional sets. Within this framework, it is natural to revisit function spaces that describe oscillatory behavior through the lens of Hausdorff content.

Building on this idea, recent works have developed analytic tools adapted to $\mathcal{H}^\beta_\infty$, including boxing inequalities **[PS20]**, maximal inequalities **[OP22; CS24; COS24; CC25]**, fractional operators **[HH26b; HKST24; HKST25; Che25; BCR25; CC26]**, and Sobolev inequalities **[HH23; HH24; HH25; CHYZ26; HH26a]**.

Motivated by the definition of BMO^β, in **[CC25]** we introduced the β-dimensional sharp maximal function and studied its basic properties in the capacitary setting. One of the main results there was the proof of a Fefferman–Stein inequality with respect to Hausdorff content, which required developing good-lambda methods in a framework where the underlying set function is non-additive. These ideas also led to a capacitary analogue of the Muckenhoupt–Wheeden inequality. Later, in **[CC26]** we applied the tools developed in the previous work to the study of commutators of fractional integrals, obtaining a characterization of BMO^β in terms of the boundedness of the Riesz commutator $[b, I_\alpha]$ in capacitary Lebesgue spaces. Since the material in this chapter is somewhat less directly connected to the main line of the thesis, we have chosen to include a separate preliminary section in order to make the exposition as self-contained as possible.

6.2 Preliminaries

In this section, we recall some basic facts concerning the Choquet integral with respect to Hausdorff content and dyadic Hausdorff content. Most of these results can be found in **[CS24]**. For a comprehensive and well-written introduction to the subject, we also refer the reader to **[PS23]**. Although the arguments

in this chapter follow the Choquet-integral approach, there are other useful ways to build function spaces over outer measures. In particular, Do and Thiele developed an L^p theory for outer measure spaces, based on sizes and adapted to Carleson embeddings and time-frequency analysis **[DT15]**.

6.2.1 Capacities, Choquet integrals, and dyadic Hausdorff content

We begin by recalling the notion of outer measure.

Definition 6.1. A set function $C : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ is called an outer measure if it satisfies the following three properties:

(i) *Empty set:*

$$C(\emptyset) = 0.$$

(ii) *Monotonicity:* if $E \subset F \subset \mathbb{R}^n$, then

$$C(E) \leq C(F).$$

(iii) *Countable subadditivity:* for every sequence $\{E_j\}_{j=1}^{\infty} \subset \mathcal{P}(\mathbb{R}^n)$,

$$C\Big(\bigcup_{j=1}^{\infty} E_j\Big) \leq \sum_{j=1}^{\infty} C(E_j).$$

In the literature, such a set function is also called a *capacity in the sense of N. Meyers*. A set function C satisfying the preceding properties and the following two properties is called a *capacity in the sense of Choquet*:

(iv) *Outer regularity:* for every non-increasing sequence of compact sets $\{K_j\}_j \subset \mathcal{P}(\mathbb{R}^n)$,

$$C\left(\bigcap_{j=0}^{\infty} K_j\right) = \lim_{j\to\infty} C(K_j)$$

(v) *Inner regularity:* for every nondecreasing sequence of sets $\{E_j\}_j \subset \mathcal{P}(\mathbb{R}^n)$,

$$C\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} C(E_j).$$

Before introducing the Choquet integral, we also recall the notion of quasicontinuity associated with a capacity.

Definition 6.2. Let $E \subseteq \mathbb{R}^n$. A function $f : E \to [-\infty, \infty]$ is said to be *C-quasicontinuous* if for every $\varepsilon > 0$ there exists an open set $G \subseteq \mathbb{R}^n$ such that $C(G) < \varepsilon$ and the restriction $f|_{E\setminus G}$ is continuous on $E \setminus G$ with respect to the relative topology.

We now recall the definition of the Choquet integral with respect to an outer measure C. Thus, if C is an outer measure, the Choquet integral of a non-negative function f over a set $\Omega \subseteq \mathbb{R}^n$ is defined by

$$\int_\Omega f \, dC := \int_0^\infty C(\{x \in \Omega : f(x) > t\}) \, dt. \tag{6.1}$$

Here the right-hand side is understood as the Lebesgue integral on $[0, \infty)$ of the function

$$t \mapsto C(\{x \in \Omega : f(x) > t\}),$$

which is well defined since this function is decreasing. This integral is positively 1-homogeneous and monotone. In general, the Choquet integral is not linear, and it may fail even to be sublinear. In **[Cho54]**, Choquet showed that for capacities in the sense of Choquet, sublinearity is equivalent to the following additional property of the set function.

Definition 6.3. A set function $C : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ is said to be *strongly subadditive* if

$$C(E \cap F) + C(E \cup F) \leq C(E) + C(F),$$

for every $E, F \subset \mathbb{R}^n$.

Theorem 6.4. *Let C be a capacity in the sense of Choquet. Then the Choquet integral* (6.1) *is sublinear if and only if C is strongly subadditive.*

In the case $\beta < n$, the set function $\mathcal{H}^\beta_\infty$ is not strongly subadditive, and therefore the associated Choquet integral is not sublinear by Theorem 6.4. Nevertheless, it still satisfies the basic estimates recorded in the next lemma. The following result can be found in **[HH23, p. 5]**, with its proof referred to in **[Ada98]** and **[Ada15, Chapter 4]**.

Lemma 6.5. *Let $0 < \beta \leq n$ and let $\Omega \subseteq \mathbb{R}^n$. Then the following hold:*

(i) For every $a \geq 0$ and every non-negative f,

$$\int_\Omega a\, f \, d\mathcal{H}^\beta_\infty = a \int_\Omega f \, d\mathcal{H}^\beta_\infty.$$

(ii) For non-negative functions f_1, f_2,

$$\int_\Omega (f_1 + f_2) \, d\mathcal{H}^\beta_\infty \leq 2 \left(\int_\Omega f_1 \, d\mathcal{H}^\beta_\infty + \int_\Omega f_2 \, d\mathcal{H}^\beta_\infty \right).$$

(iii) If $1 < p < \infty$ and p' is the conjugate exponent, then for non-negative functions f_1, f_2,

$$\int_\Omega f_1 f_2 \, d\mathcal{H}^\beta_\infty \leq 2 \left(\int_\Omega f_1^p \, d\mathcal{H}^\beta_\infty \right)^{\frac{1}{p}} \left(\int_\Omega f_2^{p'} \, d\mathcal{H}^\beta_\infty \right)^{\frac{1}{p'}}.$$

Proof. The first property follows directly from the definition of the Choquet integral. Indeed, if $a = 0$ there is nothing to prove, while for $a > 0$ the change of variables $s = t/a$ gives

$$\int_\Omega af \, d\mathcal{H}^\beta_\infty = \int_0^\infty \mathcal{H}^\beta_\infty(\{x \in \Omega : \, af(x) > t\}) \, dt = a \int_\Omega f \, d\mathcal{H}^\beta_\infty.$$

We prove now the second property. For every $t > 0$ we have the inclusion

$$\{x \in \Omega : \ f_1(x) + f_2(x) > t\} \subset \{x \in \Omega : \ f_1(x) > t/2\} \cup \{x \in \Omega : \ f_2(x) > t/2\}.$$

Using the subadditivity of $\mathcal{H}^\beta_\infty$ and then the change of variables $s = t/2$, we obtain

$$\begin{aligned}\int_\Omega (f_1 + f_2)\, d\mathcal{H}^\beta_\infty &\le \int_0^\infty \mathcal{H}^\beta_\infty(\{x \in \Omega : \ f_1(x) > t/2\})\, dt \\ &\quad + \int_0^\infty \mathcal{H}^\beta_\infty(\{x \in \Omega : \ f_2(x) > t/2\})\, dt \\ &= 2\int_\Omega f_1\, d\mathcal{H}^\beta_\infty + 2\int_\Omega f_2\, d\mathcal{H}^\beta_\infty.\end{aligned}$$

This proves (ii). The proof of (iii) follows the classical proof of Hölder's inequality, with the extra factor 2 from the quasi-sublinearity established in (ii). □

Iterating the estimate in Lemma 6.5(ii) does not yield a countable subadditivity estimate with a uniform constant, since the constant grows with the number of summands. This is one of the reasons for introducing the dyadic Hausdorff content.

Definition 6.6. Let $0 < \beta \le n$ and let Q be a cube in $\mathbb{R}^n$. For every $E \subseteq \mathbb{R}^n$, the dyadic Hausdorff content $\mathcal{H}^{\beta,Q}_\infty(E)$ is defined by

$$\mathcal{H}^{\beta,Q}_\infty(E) := \inf\left\{\sum_{i=1}^\infty \ell(Q_i)^\beta : E \subseteq \bigcup_{i=1}^\infty Q_i,\ Q_i \in \mathcal{D}_Q\right\}, \tag{6.2}$$

where $\mathcal{D}_Q$ denotes the collection of dyadic cubes generated by Q.

Although $\mathcal{H}^\beta_\infty$ fails to be strongly subadditive when $\beta < n$, the dyadic content $\mathcal{H}^{\beta,Q}_\infty$ is strongly subadditive for every cube Q. Consequently, for every sequence of non-negative functions $\{f_j\}_{j=1}^\infty$,

$$\int_{\mathbb{R}^n} \sum_{j=1}^\infty f_j\, d\mathcal{H}^{\beta,Q}_\infty \le \sum_{j=1}^\infty \int_{\mathbb{R}^n} f_j\, d\mathcal{H}^{\beta,Q}_\infty \tag{6.3}$$

(see **[STW16, Propositions 3.5 and 3.6]** for the usual dyadic case and **[CS24, Propositions 2.6 and 2.10]** for a general cube Q). Thus, a convenient way to handle Choquet integration with respect to $\mathcal{H}^\beta_\infty$ is to pass through the dyadic Hausdorff content associated with a fixed cube. More precisely, there exists a constant $C_\beta > 0$ such that

$$\frac{1}{C_\beta}\,\mathcal{H}^{\beta,Q}_\infty(E) \le \mathcal{H}^\beta_\infty(E) \le C_\beta\,\mathcal{H}^{\beta,Q}_\infty(E), \tag{6.4}$$

for every cube Q and every set $E \subset \mathbb{R}^n$ (see **[YY08, Proposition 2.3]** and **[CS24, Proposition 2.11]**). Combining (6.4) with (6.3), we obtain

$$\int_{\mathbb{R}^n} \sum_{j=1}^\infty f_j\, d\mathcal{H}^\beta_\infty \le C_\beta^2 \sum_{j=1}^\infty \int_{\mathbb{R}^n} f_j\, d\mathcal{H}^\beta_\infty, \tag{6.5}$$

for every sequence of non-negative functions $\{f_j\}_{j=1}^\infty$.

We next record a lemma concerning the packing condition of dyadic cubes, which is a small modification of **[CS24, Proposition 2.11]** and **[OV98, Lemma 2]**. This type of packing covering lemma can be traced back to Melnikov (see **[Mel69]**).

Lemma 6.7. *Suppose that $\{Q_j\}$ is a family of pairwise disjoint dyadic cubes subordinate to a dyadic lattice $\mathcal{D}(Q_0)$. Then there exists a subfamily $\{Q_{j_k}\}$ and a family of pairwise disjoint ancestors $\{\widetilde{Q}_k\}$ such that*

1.
$$\bigcup_j Q_j \subset \left(\bigcup_k Q_{j_k}\right) \cup \left(\bigcup_k \widetilde{Q}_k\right).$$

2.
$$\sum_{Q_{j_k}\subset Q} \ell(Q_{j_k})^\beta \leq 2\,\ell(Q)^\beta \qquad \text{for every dyadic cube } Q.$$

3.
$$\begin{aligned}\mathcal{H}^{\beta,Q_0}_\infty\Big(\bigcup_j Q_j\Big) &\leq \sum_{k:\ Q_{j_k}\not\subseteq \widetilde{Q}_m} \ell(Q_{j_k})^\beta + \sum_k \ell(\widetilde{Q}_k)^\beta \\ &\leq \sum_k \ell(Q_{j_k})^\beta \\ &\leq 2\,\mathcal{H}^{\beta,Q_0}_\infty\Big(\bigcup_j Q_j\Big).\end{aligned}$$

Proof. An application of **[CS24, Proposition 2.11]** yields a subfamily $\{Q_{j_k}\}_k$ of $\{Q_j\}_j$ and a family of non-overlapping ancestors $\{\widetilde{Q}_k\}_k$ which satisfy the properties (1), (2) and

$$\ell(\widetilde{Q}_k)^\beta \leq \sum_{Q_{j_i}\subset \widetilde{Q}_k} \ell(Q_{j_i})^\beta \text{ for each } \widetilde{Q}_k.$$

Moreover, using the subadditivity of $\mathcal{H}^{\beta,Q_0}_\infty$, we obtain

$$\begin{aligned}\mathcal{H}^{\beta,Q_0}_\infty(\cup_j Q_j) &\leq \sum_{k:Q_{j_k}\not\subseteq \widetilde{Q}_m} \ell(Q_{j_k})^\beta + \sum_k \ell(\widetilde{Q}_k)^\beta \\ &\leq \sum_{k:Q_{j_k}\not\subseteq \widetilde{Q}_m} \ell(Q_{j_k})^\beta + \sum_{k:Q_{j_k}\subseteq \widetilde{Q}_m} \ell(Q_{j_k})^\beta \\ &= \sum_k \ell(Q_{j_k})^\beta. \end{aligned} \tag{6.6}$$

Now, it is left to show that

$$\sum_k \ell(Q_{j_k})^\beta \leq 2\mathcal{H}^{\beta,Q_0}_\infty(\cup_j Q_j). \tag{6.7}$$

To this end, we let $\{Q_m\}_m \subseteq \mathcal{D}(Q_0)$ be a maximal collection of dyadic cubes such that

$$\bigcup_j Q_j \subseteq \bigcup_m Q_m.$$

The property (2) then yields that

$$\sum_k \ell(Q_{j_k})^\beta = \sum_m \sum_{k:Q_{j_k}\subseteq Q_m} \ell(Q_{j_k})^\beta \leq \sum_m 2\ell(Q_m)^\beta$$

and (6.7) follows by taking the infimum over all such families of dyadic cubes $\{Q_m\}_m$. □

Remark 6.8. Condition (2) in Lemma 6.7 is usually referred to as a packing condition. One useful consequence is that if $\{Q_k\}_k \subset \mathcal{D}(Q_0)$ satisfies

$$\sum_{Q_k\subset Q} \ell(Q_k)^\beta \leq 2\,\ell(Q)^\beta \qquad \text{for every dyadic cube } Q,$$

then for every $f \geq 0$ we have

$$\sum_k \int_{Q_k} f\, d\mathcal{H}^{\beta,Q_0}_\infty \leq 2\int_{\bigcup_k Q_k} f\, d\mathcal{H}^{\beta,Q_0}_\infty. \tag{6.8}$$

We refer to **[BCRS25, Proposition 2.8]** for the proof of this property.

We shall make use of the Morrey space of measures. For $0 < \beta \leq n$, we define

$$\mathscr{M}^\beta(\mathbb{R}^n) := \left\{\mu \in M_{\mathrm{loc}}(\mathbb{R}^n) : \|\mu\|_{\mathscr{M}^\beta(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\mu(B(x,r))}{r^\beta} < \infty\right\}, \tag{6.9}$$

where $M_{\mathrm{loc}}(\mathbb{R}^n)$ denotes the space of locally finite Radon measures on $\mathbb{R}^n$. It has been shown in **[Ada88]** that the topological dual space of $L^1(\mathbb{R}^n, \mathcal{H}^\beta_\infty)$ can be identified with the Morrey space $\mathscr{M}^\beta(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} f\, d\mathcal{H}^\beta_\infty \simeq \sup_{\|\mu\|_{\mathscr{M}^\beta(\mathbb{R}^n)}\leq 1} \int_{\mathbb{R}^n} f\, d\mu \tag{6.10}$$

holds for every non-negative $\mathcal{H}^\beta_\infty$-quasicontinuous function $f \in L^1(\mathbb{R}^n, \mathcal{H}^\beta_\infty)$.

6.2.2 Dyadic maximal operators and Calderón–Zygmund decomposition

We now introduce the non-centered β-dimensional maximal operator with respect to the Hausdorff content, which was defined and studied in **[COS24]**, as well as its dyadic variant introduced in **[CS24]**.

Definition 6.9. Let $0 < \beta \leq n$. The maximal operator associated with $\mathcal{H}^\beta_\infty$ is defined by

$$\mathcal{M}_{\mathcal{H}^\beta_\infty} f(x) := \sup_{Q\ni x} \frac{1}{\ell(Q)^\beta} \int_Q |f|\, d\mathcal{H}^\beta_\infty,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes containing x. Furthermore, given a fixed cube Q_0 in $\mathbb{R}^n$, the corresponding dyadic maximal operator associated with $\mathcal{H}^{\beta,Q_0}_\infty$ is defined by

$$\mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) := \sup_{\substack{Q\in\mathcal{D}_{Q_0}\\ Q\ni x}} \frac{1}{\ell(Q)^\beta} \int_Q |f|\, d\mathcal{H}^{\beta,Q_0}_\infty.$$

A weak-type $(1,1)$ estimate for the dyadic maximal operator was established in **[CS24, Theorem 3.1]**: there exists $C = C(n,\beta) > 0$ such that

$$\mathcal{H}_\infty^{\beta,Q_0}\big(\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0} f}(x) > t\}\big) \le \frac{C}{t}\int_{\mathbb{R}^n} |f|\, d\mathcal{H}_\infty^{\beta,Q_0}, \tag{6.11}$$

for all $t > 0$ and every function f such that the right-hand side is finite.

Remark 6.10. Note that in **[CS24, Theorem 3.1]**, f is assumed to be $\mathcal{H}_\infty^\beta$-quasicontinuous. However, one finds that the weaker assumption

$$\int_{\mathbb{R}^n} |f|\, d\mathcal{H}_\infty^{\beta,Q_0} < \infty$$

is sufficient for the weak-type estimate (6.11) by following the original proof. That is, the assumption that f be quasicontinuous is not really necessary for (6.11).

The following comparison lemma between the level sets of the maximal function and its dyadic version is taken from **[COS24, Lemma 2.4]**.

Lemma 6.11. *Let Q_0 be a cube in $\mathbb{R}^n$. Then there exist constants $C, \widetilde{C} > 0$, depending only on n and β, such that*

$$\mathcal{H}_\infty^\beta\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\beta} f(x) > t\right\}\right) \le C\,\mathcal{H}_\infty^\beta\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0}} f(x) > \widetilde{C}\,2^{-(\beta+n)}t\right\}\right),$$

for every function f.

The next result refines the dyadic Calderón–Zygmund decomposition with respect to Hausdorff content, originally obtained in **[CS24, Theorem 3.4]**. Notably, we do *not* require f to be $\mathcal{H}_\infty^\beta$-quasicontinuous, a condition assumed in **[CS24]**. Our refined version of the theorem will be a key ingredient in the good-lambda arguments developed in the next sections, and we include a complete proof.

Theorem 6.12. *Let Q_0 be a cube and $\lambda > 0$. If*

$$\mathcal{H}_\infty^{\beta,Q_0}(\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0} f}(x) > \lambda\}) < \infty,$$

then there exists a countable collection of non-overlapping dyadic cubes $\{Q_k\}$ subordinate to Q_0 such that

(i) $\bigcup_k Q_k = \{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0} f}(x) > \lambda\}$,

(ii) $\{Q_k\}$ *is the maximal subcollection of all dyadic cubes contained in* $\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0} f}(x) > \lambda\}$,

(iii) $\lambda < \frac{1}{\ell(Q_k)^\beta}\int_{Q_k} |f|\, d\mathcal{H}_\infty^{\beta,Q_0} \le 2^\beta\lambda$ *for each k,*

(iv) $|f(x)| \le \lambda$ *for $\mathcal{H}^\beta$-a.e. $x \notin \bigcup_k Q_k$.*

Before proving Theorem 6.12, we need two auxiliary results regarding the local behavior of Hausdorff content. The first lemma, stated below, is taken from **[CS24, Lemma 3.2]**.

Lemma 6.13. *Let Q_0 be a cube in $\mathbb{R}^n$, let $E \subset \mathbb{R}^n$, and let $0 < \beta \le n$. Then*

$$\limsup_{Q \to x} \frac{\mathcal{H}_\infty^{\beta,Q_0}(E \cap Q)}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} = 1 \quad \text{for } \mathcal{H}^\beta\text{-almost every } x \in E,$$

where the limit is taken over all dyadic subcubes $Q \in \mathcal{D}_{Q_0}$ containing x with $\ell(Q) \to 0$.

Next, we give a result analogous to the classical Lebesgue differentiation theorem for Hausdorff content without assuming either measurability or quasicontinuity of the function. The proof essentially follows from a close reading of **[CS24, p. 987]**, but we still include it here for the sake of completeness.

Theorem 6.14. *Let $0 < \beta \le n$, let $Q_0 \subseteq \mathbb{R}^n$ be a cube, and let $f : \mathbb{R}^n \to [0, \infty)$ be a function. Then for $\mathcal{H}^\beta$-a.e. $x \in \mathbb{R}^n$, we have*

$$\limsup_{Q \to x} \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} \int_Q f \, d\mathcal{H}_\infty^{\beta,Q_0} \ge f(x), \tag{6.12}$$

where the limit on the left-hand side is taken over all dyadic cubes $Q \in \mathcal{D}_{Q_0}$ containing x as $\ell(Q) \to 0$.

Proof. For every $t > 0$, we observe that we have the inclusion

$$\begin{aligned} &\left\{ x \in \mathbb{R}^n : \limsup_{Q \to x} \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} \int_Q f \, d\mathcal{H}_\infty^{\beta,Q_0} < f(x) \right\} \\ &\qquad \subset \bigcup_{t \in \mathbb{Q}^+} \left\{ x \in \mathbb{R}^n : \limsup_{Q \to x} \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} \int_Q f \, d\mathcal{H}_\infty^{\beta,Q_0} < t < f(x) \right\}, \end{aligned} \tag{6.13}$$

where $\mathbb{Q}^+$ denotes the set of all positive rational numbers. Moreover, for every $t \in \mathbb{Q}^+$, define

$$E_t := \{ x \in \mathbb{R}^n : f(x) > t \}.$$

By Lemma 6.13, for $\mathcal{H}^\beta$-almost every $x \in E_t$, we have

$$\begin{aligned} \limsup_{Q \to x} \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} \int_Q f \, d\mathcal{H}_\infty^{\beta,Q_0} &\ge \limsup_{Q \to x} \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} \int_{Q \cap E_t} t \, d\mathcal{H}_\infty^{\beta,Q_0} \\ &= t \limsup_{Q \to x} \frac{\mathcal{H}_\infty^{\beta,Q_0}(Q \cap E_t)}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} \\ &= t. \end{aligned}$$

That is, the set

$$\begin{aligned} &E_t \setminus \left\{ x \in \mathbb{R}^n : \limsup_{Q \to x} \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} \int_Q f \, d\mathcal{H}_\infty^{\beta,Q_0} \ge f(x) \right\} \\ &\qquad = \left\{ x \in \mathbb{R}^n : \limsup_{Q \to x} \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q)} \int_Q f \, d\mathcal{H}_\infty^{\beta,Q_0} < t < f(x) \right\} \end{aligned}$$

has $\mathcal{H}^\beta$-capacity zero. It then follows from (6.13) that

$$\begin{aligned}
&\mathcal{H}^\beta_\infty\left(\left\{x\in\mathbb{R}^n : \limsup_{Q\to x}\frac{1}{\mathcal{H}^{\beta,Q_0}_\infty(Q)}\int_Q f\,d\mathcal{H}^{\beta,Q_0}_\infty < f(x)\right\}\right)\\
&\quad\le \sum_{t\in\mathbb{Q}^+}\mathcal{H}^\beta_\infty\left(\left\{x\in\mathbb{R}^n : \limsup_{Q\to x}\frac{1}{\mathcal{H}^{\beta,Q_0}_\infty(Q)}\int_Q f\,d\mathcal{H}^{\beta,Q_0}_\infty < t < f(x)\right\}\right)\\
&\quad= 0,
\end{aligned}$$

which completes the proof. □

We are now ready to give the proof of Theorem 6.12.

Proof of Theorem 6.12. Let $\{Q_i\}_i$ be the collection of all dyadic cubes contained in $\{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > \lambda\}$. Since $\mathcal{D}_{Q_0}$ is countable and

$$\mathcal{H}^{\beta,Q_0}_\infty(\{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > \lambda\}) < \infty,$$

the maximal subcollection of $\{Q_i\}_i$ exists and we denote it by $\{Q_k\}_k$. In particular, we have

$$\bigcup_k Q_k = \{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > \lambda\}.$$

and thus

$$\lambda < \frac{1}{\ell(Q_k)^\beta}\int_{Q_k}|f|\,d\mathcal{H}^{\beta,Q_0}_\infty.$$

Now let $\widetilde{Q}_k$ denote the parent of Q_k for each k. Using the fact that $\widetilde{Q}_k$ is not contained in $\{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > \lambda\}$, we have

$$\frac{1}{2^\beta\ell(Q_k)^\beta}\int_{Q_k}|f|\,d\mathcal{H}^{\beta,Q_0}_\infty \le \frac{1}{\ell(\widetilde{Q}_k)^\beta}\int_{\widetilde{Q}_k}|f|\,d\mathcal{H}^{\beta,Q_0}_\infty \le \lambda.$$

Finally, (iv) follows by Theorem 6.14. This completes the proof. □

6.2.3 BMO$^\beta$ and basic properties

We now recall the definition of the space BMO$^\beta$, introduced in **[CS24]**, together with the John–Nirenberg inequality in this space.

Definition 6.15. A function f is said to belong to $\mathrm{BMO}^\beta(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} := \sup_Q \inf_{c\in\mathbb{R}}\frac{1}{\ell(Q)^\beta}\int_Q |f-c|\,d\mathcal{H}^\beta_\infty < +\infty. \tag{6.14}$$

In **[CS24]**, it is further shown that any function $f\in\mathrm{BMO}^\beta(\mathbb{R}^n)$ satisfies a John–Nirenberg-type estimate with respect to the Hausdorff content. Namely, there exist constants $c_\beta, C_\beta > 0$ such that for every cube Q, for all $t>0$, and for a suitably chosen $c_Q\in\mathbb{R}$, we have

$$\mathcal{H}^\beta_\infty\big(\{x\in Q : |f(x)-c_Q| > t\}\big) \le C_\beta\,\ell(Q)^\beta\exp\left(-\frac{c_\beta\,t}{\|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}}\right). \tag{6.15}$$

We also recall the following result from **[CS24, Corollary 1.5]**.

Theorem 6.16. *Let $0 < \beta \le n \in \mathbb{N}$ and $f \in \mathrm{BMO}^\beta(\mathbb{R}^n)$. There exists a constant $C = C(\beta) > 0$ such that*

$$\sup_Q \inf_{c\in\mathbb{R}} \left(\frac{1}{\ell(Q)^\beta} \int_Q |f-c|^p \, d\mathcal{H}^\beta_\infty \right)^{\frac{1}{p}} \le Cp \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)},$$

for all $p \ge 1$.

For a function $f \in \mathrm{BMO}^\beta(\mathbb{R}^n)$ and any cube Q, we define the quantity

$$f_Q = \arg\min_{c\in\mathbb{R}} \frac{1}{\ell(Q)^\beta} \int_Q |f-c| \, d\mathcal{H}^\beta_\infty.$$

Although in the previous chapters the notation f_Q was used to denote the Lebesgue average of f over Q, from now on, in this chapter, f_Q will denote the above minimizing constant. We observe that, for every $f \in \mathrm{BMO}^\beta(\mathbb{R}^n)$ and every cube Q, the infimum in the definition of $\|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}$ is always attained; that is, the previous arg min exists. See **[CS24, p. 983]** for a detailed discussion.

The next lemma records a control of these quantities between nested cubes.

Lemma 6.17. *Let $0 < \beta \le n \in \mathbb{N}$ and $f \in \mathrm{BMO}^\beta(\mathbb{R}^n)$. Then, there exists $C_\beta > 0$ such that*

$$|f_{2^k Q} - f_Q| \le C_\beta k \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)},$$

for each cube Q and each $k \in \mathbb{N}$.

Proof. Firstly, we observe that

$$\begin{aligned}
|f_{2Q} - f_Q| =& \frac{C_\beta}{\ell(Q)^\beta} \int_Q |f_{2Q} - f_Q| d\mathcal{H}^\beta_\infty \\
\le& \frac{C_\beta}{\ell(Q)^\beta} \int_Q |f - f_{2Q}| d\mathcal{H}^\beta_\infty + \frac{C_\beta}{\ell(Q)^\beta} \int_Q |f - f_Q| d\mathcal{H}^\beta_\infty \\
\le& \frac{C_\beta}{\ell(2Q)^\beta} \int_{2Q} |f - f_{2Q}| d\mathcal{H}^\beta_\infty + \frac{C_\beta}{\ell(Q)^\beta} \int_Q |f - f_Q| d\mathcal{H}^\beta_\infty \\
\le& C_\beta \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|f_{2^k Q} - f_Q| \le& \sum_{i=1}^k |f_{2^i Q} - f_{2^{i-1} Q}| \\
\le& \sum_{i=1}^k C_\beta \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \\
=& C_\beta k \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}.
\end{aligned}$$

□

We can equivalently state the John–Nirenberg inequality (6.15) in terms of exponential integrability, as shown in the following result.

Theorem 6.18. *Let $0 < \beta \le n \in \mathbb{N}$ and $f \in \mathrm{BMO}^\beta(\mathbb{R}^n)$. There exist constants $C_1, C_2 > 0$ such that*

$$\frac{1}{\ell(Q)^\beta} \int_Q \exp\left(\frac{|f - f_Q|}{C_1 \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}}\right) d\mathcal{H}^\beta_\infty \le C_2,$$

for each cube Q.

We now collect several elementary properties of BMO$^\beta$ that will be used repeatedly later.

Lemma 6.19. *Let $0 < \beta \le n \in \mathbb{N}$. Then there exists a constant $C_\beta > 0$ depending only on β such that*

(i) $\|f + g\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \le 2(\|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} + \|g\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)})$.

(ii) $\|\lambda f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} = |\lambda| \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}$ *for any* $\lambda \in \mathbb{R}$.

(iii) $\||f|\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \le C_\beta \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}$.

(iv) $\|\max(f, g)\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \le C_\beta(\|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} + \|g\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)})$.

(v) $\|\min(f, g)\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \le C_\beta(\|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} + \|g\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)})$.

Proof. The proof of (i) and (ii) follows immediately from Lemma 6.5.

We now prove (iii). The argument is a slight modification of **[BCR25, Lemma 2.6]**, and we include it for completeness.

We first consider the dyadic version of the BMO$^\beta$ seminorm. Let

$$\|f\|_{\mathrm{BMO}^\beta(\mathcal{H}^{\beta,Q_0}_\infty)} := \sup_{Q \subseteq \mathbb{R}^n} \inf_{c \in \mathbb{R}} \frac{1}{\mathcal{H}^{\beta,Q_0}_\infty(Q)} \int_Q |f(y) - c| \, d\mathcal{H}^{\beta,Q_0}_\infty(y),$$

where $Q_0 := [0, 1)^n$. By (6.4), this dyadic seminorm is equivalent to the original one:

$$\|f\|_{\mathrm{BMO}^\beta(\mathcal{H}^{\beta,Q_0}_\infty)} \simeq \|f\|_{\mathrm{BMO}^\beta}.$$

Now fix a cube $Q' \subseteq \mathbb{R}^n$, and choose

$$c_{Q'} := \arg\min_{c \in \mathbb{R}} \frac{1}{\mathcal{H}^{\beta,Q_0}_\infty(Q')} \int_{Q'} |f - c| \, d\mathcal{H}^{\beta,Q_0}_\infty.$$

We also set

$$f_{Q',\beta} := \frac{1}{\mathcal{H}^{\beta,Q_0}_\infty(Q')} \int_{Q'} |f| \, d\mathcal{H}^{\beta,Q_0}_\infty.$$

By the dyadic averaging estimate, see **[BCR25, Lemma 2.3]**, we have

$$|f_{Q',\beta} - c| \le \frac{1}{\mathcal{H}^{\beta,Q_0}_\infty(Q')} \int_{Q'} \big||f| - c\big| \, d\mathcal{H}^{\beta,Q_0}_\infty,$$

for every $c \geq 0$. In particular, taking $c = |f(x)|$ for $x \in Q'$, we obtain

$$\begin{aligned}
\big||f(x)| - f_{Q',\beta}\big| \leq & \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q')} \int_{Q'} \big||f(x)| - |f(y)|\big| \, d\mathcal{H}_\infty^{\beta,Q_0}(y) \\
\leq & \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q')} \int_{Q'} |f(x) - f(y)| \, d\mathcal{H}_\infty^{\beta,Q_0}(y) \\
\leq & \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q')} \int_{Q'} |f(x) - c_{Q'}| \, d\mathcal{H}_\infty^{\beta,Q_0}(y) \\
& + \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q')} \int_{Q'} |f(y) - c_{Q'}| \, d\mathcal{H}_\infty^{\beta,Q_0}(y) \\
= & |f(x) - c_{Q'}| + \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q')} \int_{Q'} |f(y) - c_{Q'}| \, d\mathcal{H}_\infty^{\beta,Q_0}(y) \\
\leq & |f(x) - c_{Q'}| + \|f\|_{\mathrm{BMO}^\beta(\mathcal{H}_\infty^{\beta,Q_0})}.
\end{aligned}$$

We now integrate this inequality over Q' with respect to $\mathcal{H}_\infty^\beta$, divide by $\mathcal{H}_\infty^\beta(Q')$, and use again (6.4). This gives

$$\begin{aligned}
\frac{1}{\mathcal{H}_\infty^\beta(Q')} \int_{Q'} \big||f(x)| - f_{Q',\beta}\big| \, d\mathcal{H}_\infty^\beta(x) \leq & \frac{1}{\mathcal{H}_\infty^\beta(Q')} \int_{Q'} |f(x) - c_{Q'}| \, d\mathcal{H}_\infty^\beta(x) + \|f\|_{\mathrm{BMO}^\beta(\mathcal{H}_\infty^{\beta,Q_0})} \\
\leq & C_\beta \frac{1}{\mathcal{H}_\infty^{\beta,Q_0}(Q')} \int_{Q'} |f(x) - c_{Q'}| \, d\mathcal{H}_\infty^{\beta,Q_0}(x) \\
& + \|f\|_{\mathrm{BMO}^\beta(\mathcal{H}_\infty^{\beta,Q_0})} \\
\leq & C_\beta \|f\|_{\mathrm{BMO}^\beta(\mathcal{H}_\infty^{\beta,Q_0})}.
\end{aligned}$$

Since $f_{Q',\beta}$ is just a particular choice of constant, it follows that

$$\inf_{c \in \mathbb{R}} \frac{1}{\mathcal{H}_\infty^\beta(Q')} \int_{Q'} ||f(x)| - c| \, d\mathcal{H}_\infty^\beta(x) \leq C_\beta \|f\|_{\mathrm{BMO}^\beta(\mathcal{H}_\infty^{\beta,Q_0})} \leq C_\beta \|f\|_{\mathrm{BMO}^\beta}.$$

Taking the supremum over all cubes $Q' \subset \mathbb{R}^n$, we conclude that

$$\||f|\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \leq C_\beta \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}.$$

Finally, (iv) and (v) follow from (i), (ii), and (iii) together with the identities

$$\max(f,g) = \frac{f + g + |f - g|}{2}, \qquad \min(f,g) = \frac{f + g - |f - g|}{2}.$$

□

As an immediate consequence of the previous lemma, we obtain the stability of BMO^β under truncation.

Lemma 6.20. *Let $0 < \beta \leq n$, $f \in \mathrm{BMO}^\beta(\mathbb{R}^n)$ and $K > 0$. Let the truncated function $\tau_K f$ be defined as*

$$\tau_K f(x) := \begin{cases} K, & \text{if } f(x) > K, \\ f(x), & \text{if } -K \leq f(x) \leq K, \\ -K & \text{if } f(x) < -K. \end{cases}$$

Then there exists a constant $C_\beta > 0$, depending only on β, such that

$$\|\tau_K f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \le C_\beta \|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}.$$

6.3 β-dimensional sharp maximal function

The definition of BMO$^\beta$ naturally leads to the introduction of the β-dimensional sharp maximal operator.

Definition 6.21. Let $0 < \beta \le n \in \mathbb{N}$. The β-dimensional sharp maximal operator of a function f in $\mathbb{R}^n$ is defined as

$$\mathcal{M}_\beta^\# f(x) := \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \frac{1}{\ell(Q)^\beta} \int_Q |f - c| \, d\mathcal{H}_\infty^\beta. \tag{6.16}$$

The dyadic sharp maximal operator associated with Q_0 is defined by

$$\mathcal{M}_{\beta,Q_0}^\# f(x) := \sup_{\substack{Q \in \mathcal{D}_{Q_0} \\ Q \ni x}} \inf_{c \in \mathbb{R}} \frac{1}{\ell(Q)^\beta} \int_Q |f - c| \, d\mathcal{H}_\infty^{\beta,Q_0}.$$

We also refer to **[HH26b, p. 16]** for the β-dimensional sharp maximal function defined with respect to balls in Euclidean space. By definition, we have

$$\|f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} = \|\mathcal{M}_\beta^\# f\|_{L^\infty(\mathbb{R}^n)}.$$

Moreover, it is easy to check that

$$\mathcal{M}_\beta^\# f \le \mathcal{M}_{\mathcal{H}_\infty^\beta} f.$$

The main objective of this section is to prove a capacitary Fefferman–Stein inequality comparing $\mathcal{M}_{\mathcal{H}_\infty^\beta}$ and $\mathcal{M}_\beta^\#$, which extends (1.7) to the framework of the Choquet integral with respect to Hausdorff content.

Theorem 6.22. *Let $0 < \beta \le n \in \mathbb{N}$, $0 < p_0 < \infty$ and $f \in L^1_{loc}(\mathcal{H}_\infty^\beta)$ such that*

$$\sup_{0<t\le N} t^{p_0} \mathcal{H}_\infty^\beta(\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\beta} f(x) > t\}) < \infty \qquad \text{for all } N > 0. \tag{6.17}$$

Then there exists a constant $C = C(p, n, \beta) > 0$ such that

$$\left(\int_{\mathbb{R}^n} \left(\mathcal{M}_{\mathcal{H}_\infty^\beta} f \right)^p \, d\mathcal{H}_\infty^\beta \right)^{\frac{1}{p}} \le C \left(\int_{\mathbb{R}^n} \left(\mathcal{M}_\beta^\# f \right)^p \, d\mathcal{H}_\infty^\beta \right)^{\frac{1}{p}}, \tag{6.18}$$

for all $p_0 < p < \infty$, and also

$$\left\| \mathcal{M}_{\mathcal{H}_\infty^\beta} f \right\|_{L^{p,\infty}(\mathcal{H}_\infty^\beta)} \le C \left\| \mathcal{M}_\beta^\# f \right\|_{L^{p,\infty}(\mathcal{H}_\infty^\beta)}, \tag{6.19}$$

for all $p_0 \le p < \infty$.

Here, $L^{p,\infty}(\mathcal{H}_\infty^\beta)$ denotes the vector space of all the functions $g : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\|g\|_{L^{p,\infty}(\mathcal{H}_\infty^\beta)} := \sup_{\lambda>0} \lambda \, \mathcal{H}_\infty^\beta(\{x \in \mathbb{R}^n : |g(x)| > \lambda\})^{\frac{1}{p}} < \infty.$$

Remark 6.23. The condition (6.17) is quite general, and it is satisfied, for example, for each function f such that

$$\int_{\mathbb{R}^n} |f|^{p_0} d\mathcal{H}^\beta_\infty < \infty,$$

with $p_0 \geq 1$, by the weak type (p,p) inequality satisfied by $\mathcal{M}_{\mathcal{H}^\beta_\infty}$ (see **[COS24]**). We emphasize that in the previous result, we do not require the function f to be measurable; the result holds for non-measurable functions. In the case $\beta = n$, we recover (1.7) while extending it to the setting of non-measurable functions.

The key ingredient in the proof of the previous result is a dyadic good-lambda inequality. In **[FS72]**, Fefferman and Stein proved (1.7) from the good-lambda estimate

$$|\{x \in \mathbb{R}^n : \mathcal{M}f > t\}| \leq |\{x \in \mathbb{R}^n : \mathcal{M}^\# f > \frac{t}{A}\}| + \frac{2}{A}|\{x \in \mathbb{R}^n : \mathcal{M}f > 2^{-n-1}t\}|, \tag{6.20}$$

for all positive t and A. In the following result we establish an analogue of (6.20) in the context of Hausdorff content. The proof follows the same general strategy; however, the nonlinearity of the Choquet integral introduces some technical difficulties, which we overcome via the packing condition in Lemma 6.7.

Theorem 6.24. *Let $n \in \mathbb{N}$ and $0 < \beta \leq n$. Let $Q_0 \subset \mathbb{R}^n$ be a cube, and let f be a function on $\mathbb{R}^n$. Then, we have*

$$\mathcal{H}^{\beta,Q_0}_\infty\big(\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > t\}\big) \leq \mathcal{H}^{\beta,Q_0}_\infty\left(\left\{x \in \mathbb{R}^n : \mathcal{M}^\#_{\beta,Q_0} f(x) > \frac{t}{A}\right\}\right) + \frac{8}{A}\,\mathcal{H}^{\beta,Q_0}_\infty\big(\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > 2^{-\beta-2}t\}\big),$$

for every $t > 0$ and every $A > 0$.

Proof. Define $\phi : (0,\infty) \to [0,\infty]$ by

$$\phi(t) := \mathcal{H}^{\beta,Q_0}_\infty\big(\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > t\}\big).$$

With this notation, it is enough to prove that

$$\phi(t) \leq \mathcal{H}^{\beta,Q_0}_\infty\left(\left\{x \in \mathbb{R}^n : \mathcal{M}^\#_{\beta,Q_0} f(x) > \frac{t}{A}\right\}\right) + \frac{8}{A}\phi(2^{-\beta-2}t),$$

for every $t > 0$ and $A > 0$.

Fix $t > 0$. We apply Theorem 6.12 at the levels t and $2^{-\beta-2}t$. In this way, we obtain two countable collections of pairwise non-overlapping dyadic cubes, denoted by $\{Q^t_k\}_k$ and $\{Q^{2^{-\beta-2}t}_k\}_k$, such that properties (i)–(iv) in Theorem 6.12 hold with $\lambda = t$ and $\lambda = 2^{-\beta-2}t$, respectively. More precisely, the family $\{Q^t_k\}_k$ satisfies

(A_1) $\bigcup_k Q^t_k = \{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > t\}$,

(A_2) $\{Q^t_k\}_k$ is the maximal subcollection of all the dyadic cubes contained in the set $\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > t\}$,

(A_3) $t < \frac{1}{\ell(Q_k^t)^\beta} \int_{Q_k^t} |f| \, d\mathcal{H}_\infty^{\beta,Q_0} \leq 2^\beta t$,

(A_4) $|f(x)| \leq t$ for $\mathcal{H}^\beta$-a.e. $x \notin \bigcup_k Q_k^t$.

Similarly, the family $\{Q_k^{2^{-\beta-2}t}\}_k$ satisfies

(B_1) $\bigcup_k Q_k^{2^{-\beta-2}t} = \{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0}} f(x) > 2^{-\beta-2}t\}$,

(B_2) $\{Q_k^{2^{-\beta-2}t}\}_k$ is the maximal subcollection of all the dyadic cubes contained in the set $\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0}} f(x) > 2^{-\beta-2}t\}$,

(B_3) $2^{-\beta-2}t < \frac{1}{\ell(Q_k^{2^{-\beta-2}t})^\beta} \int_{Q_k^{2^{-\beta-2}t}} |f| \, d\mathcal{H}_\infty^{\beta,Q_0} \leq \frac{t}{4}$,

(B_4) $|f(x)| \leq 2^{-\beta-2}t$ for $\mathcal{H}^\beta$-a.e. $x \notin \bigcup_k Q_k^{2^{-\beta-2}t}$.

Since $t > 2^{-\beta-2}t$, every cube in $\{Q_k^t\}_k$ is contained in some cube of $\{Q_k^{2^{-\beta-2}t}\}_k$ by (A_2) and (B_2). This allows us to split the family $\{Q_k^t\}_k$ according to whether the corresponding larger cube lies entirely inside the level set of the dyadic sharp maximal function. Thus,

$$\{Q_k^t\}_k = \mathcal{A}_1 \cup \mathcal{A}_2,$$

where

$$\mathcal{A}_1 := \left\{ Q_k^t : Q_k^t \subseteq Q_{k_0}^{2^{-\beta-2}t} \text{ for some } k_0 \text{ and } Q_{k_0}^{2^{-\beta-2}t} \subseteq \{x \in \mathbb{R}^n : \mathcal{M}_{\beta,Q_0}^{\#} f(x) > \frac{t}{A}\} \right\},$$

and

$$\mathcal{A}_2 := \left\{ Q_k^t : Q_k^t \subseteq Q_{k_0}^{2^{-\beta-2}t} \text{ for some } k_0 \text{ and } Q_{k_0}^{2^{-\beta-2}t} \not\subseteq \{x \in \mathbb{R}^n : \mathcal{M}_{\beta,Q_0}^{\#} f(x) > \frac{t}{A}\} \right\}.$$

By the subadditivity of $\mathcal{H}_\infty^{\beta,Q_0}$, we then obtain

$$\phi(t) \leq \mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_{Q_k^t \in \mathcal{A}_1} Q_k^t \right) + \mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_{Q_k^t \in \mathcal{A}_2} Q_k^t \right).$$

The contribution of $\mathcal{A}_1$ is immediate, since

$$\bigcup_{Q_k^t \in \mathcal{A}_1} Q_k^t \subseteq \left\{ x : \mathcal{M}_{\beta,Q_0}^{\#} f(x) > \frac{t}{A} \right\},$$

and therefore

$$\mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_{Q_k^t \in \mathcal{A}_1} Q_k^t \right) \leq \mathcal{H}_\infty^{\beta,Q_0}\left(\left\{ x : \mathcal{M}_{\beta,Q_0}^{\#} f(x) > \frac{t}{A} \right\} \right).$$

Consequently, it remains only to estimate the part corresponding to $\mathcal{A}_2$, namely to prove that

$$\mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_{Q_k^t \in \mathcal{A}_2} Q_k^t \right) \leq \frac{8}{A} \phi(2^{-\beta-2}t).$$

To this end, let $\{Q'_k\}_k$ be the subfamily of $\{Q_k^{2^{-\beta-2}t}\}_k$ defined by

$$\left\{Q'_k\right\}_k := \left\{Q \in \{Q_k^{2^{-\beta-2}t}\}_k : Q \text{ contains some cube in } \mathcal{A}_2\right\}.$$

We now apply Lemma 6.7 to $\{Q'_k\}_k$. This yields a subfamily $\{Q'_{k_v}\}_v$ and a family of non-overlapping ancestors $\{\widetilde{Q}_k\}_k$ such that

$$\bigcup_k Q'_k \subset \bigcup_v Q'_{k_v} \cup \bigcup_k \widetilde{Q}_k,$$

$$\sum_{Q'_{k_v} \subset Q} \ell(Q'_{k_v})^\beta \le 2\ell(Q)^\beta \qquad \text{for each dyadic cube } Q \in \mathcal{D}(Q_0),$$

and

$$\begin{aligned} \mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_k Q'_k\right) &\le \sum_{v, Q'_{k_v} \not\subseteq \widetilde{Q}_k} \ell(Q'_{k_v})^\beta + \sum_k \ell(\widetilde{Q}_k)^\beta \\ &\le 2\mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_k Q'_k\right). \end{aligned} \tag{6.21}$$

Next, let $Q' \in \{Q'_{k_v}\}_v \cup \{\widetilde{Q}_k\}_k$. By construction, Q' contains a cube in $\mathcal{A}_2 \subseteq \{Q_k^{2^{-\beta-2}t}\}_k$, and hence

$$Q' \not\subseteq \left\{x : \mathcal{M}^{\#}_{\beta,Q_0} f(x) > \frac{t}{A}\right\}.$$

Therefore, using (B_2), (B_3), and the definition of $\mathcal{M}^{\#}_{\beta,Q_0} f$, we obtain

$$\frac{1}{\ell(Q')^\beta} \int_{Q'} |f| \, d\mathcal{H}_\infty^{\beta,Q_0} \le \frac{t}{4} \tag{6.22}$$

and

$$\frac{1}{\ell(Q')^\beta} \int_{Q'} |f - c_{Q'}| \, d\mathcal{H}_\infty^{\beta,Q_0} \le \min\left(\frac{t}{A}, \frac{t}{4}\right), \tag{6.23}$$

where

$$c_{Q'} := \arg\min_{c \in \mathbb{R}} \frac{1}{\ell(Q')^\beta} \int_{Q'} |f - c| \, d\mathcal{H}_\infty^{\beta,Q_0}.$$

At this point we need a bound for $c_{Q'}$. By (6.3), together with (6.22) and (6.23), we have

$$\begin{aligned} |c_{Q'}| &= \frac{1}{\ell(Q')^\beta} \int_{Q'} |c_{Q'}| \, d\mathcal{H}_\infty^{\beta,Q_0} \\ &\le \frac{1}{\ell(Q')^\beta} \int_{Q'} |c_{Q'} - f| \, d\mathcal{H}_\infty^{\beta,Q_0} + \frac{1}{\ell(Q')^\beta} \int_{Q'} |f| \, d\mathcal{H}_\infty^{\beta,Q_0} \\ &\le \frac{t}{4} + \frac{t}{4} \\ &= \frac{t}{2}. \end{aligned}$$

It follows that if $Q_k^t \in \mathcal{A}_2$ and $Q_k^t \subseteq Q'$, then

$$\begin{aligned}\int_{Q_k^t} |f(x) - c_{Q'}|\, d\mathcal{H}_\infty^{\beta,Q_0} &\geq \int_{Q_k^t} |f|\, d\mathcal{H}_\infty^{\beta,Q_0} - c_{Q'}\ell(Q_k^t)^\beta \\ &\geq t\ell(Q_k^t)^\beta - \frac{t}{2}\ell(Q_k^t)^\beta \\ &= \frac{t}{2}\ell(Q_k^t)^\beta. \end{aligned} \tag{6.24}$$

We now return to the family $\mathcal{A}_2$. Applying Lemma 6.7 and (6.6) to $\mathcal{A}_2$, we obtain a non-overlapping subfamily $\{Q_{k_j}^t\}_j$ satisfying

$$\sum_{Q_{k_j}^t \subseteq Q} \ell(Q_{k_j}^t)^\beta \leq 2\ell(Q)^\beta \qquad \text{for each dyadic cube } Q \in \mathcal{D}(Q_0) \tag{6.25}$$

and

$$\mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_{Q_k^t \in \mathcal{A}_2} Q_k^t\right) \leq \sum_j \ell(Q_{k_j}^t)^\beta. \tag{6.26}$$

Combining (6.24) and (6.26), we obtain

$$\begin{aligned}\mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_{Q_k^t \in \mathcal{A}_2} Q_k^t\right) &\leq \sum_j \ell(Q_{k_j}^t)^\beta \\ &= \sum_v \sum_{Q_{k_j}^t \subseteq Q'_{k_v}} \ell(Q_{k_j}^t)^\beta + \sum_k \sum_{Q_{k_j}^t \subseteq \widetilde{Q}_k} \ell(Q_{k_j}^t)^\beta \\ &\leq \frac{2}{t}\sum_v \sum_{Q_{k_j}^t \subseteq Q'_{k_v}} \int_{Q'_{k_v}} |f - c_{Q'_{k_v}}|\, d\mathcal{H}_\infty^{\beta,Q_0} \\ &\quad + \frac{2}{t}\sum_k \sum_{Q_{k_j}^t \subseteq \widetilde{Q}_k} \int_{\widetilde{Q}_k} |f - c_{\widetilde{Q}_k}|\, d\mathcal{H}_\infty^{\beta,Q_0} \\ &\leq \frac{4}{t}\sum_v \int_{Q'_{k_v}} |f - c_{Q'_{k_v}}|\, d\mathcal{H}_\infty^{\beta,Q_0} \\ &\quad + \frac{4}{t}\sum_k \int_{\widetilde{Q}_k} |f - c_{\widetilde{Q}_k}|\, d\mathcal{H}_\infty^{\beta,Q_0}, \end{aligned} \tag{6.27}$$

where the last inequality follows from the packing condition (6.25) of $\{Q_{k_j}^t\}_j$ and (6.8).

Finally, since $\{Q'_k\}_k$ is a subfamily of $\{Q_k^{2^{-\beta-2}t}\}_k$, we may combine (6.21), (6.23), and (6.27) to conclude that

$$\begin{aligned}\mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_{Q_k^t \in \mathcal{A}_2} Q_k^t\right) &\leq \frac{4}{t}\sum_v \int_{Q'_{k_v}} |f - c_{Q'_{k_v}}|\, d\mathcal{H}_\infty^{\beta,Q_0} \\ &\quad + \frac{4}{t}\sum_k \int_{\widetilde{Q}_k} |f - c_{\widetilde{Q}_k}|\, d\mathcal{H}_\infty^{\beta,Q_0} \\ &\leq \frac{4}{t}\frac{t}{A}\left(\sum_v \ell(Q'_{k_v})^\beta + \sum_k \ell(\widetilde{Q}_k)^\beta\right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{8}{A}\mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_k Q_k'\right)\\
&\leq \frac{8}{A}\mathcal{H}_\infty^{\beta,Q_0}\left(\bigcup_k Q_k^{2^{-\beta-2}t}\right)\\
&= \frac{8}{A}\phi(2^{-\beta-2}t),
\end{aligned}$$

as required. □

We now prove Theorem 6.22. The argument combines the comparison between the global and dyadic maximal operators, the good-lambda inequality from Theorem 6.24, and the layer-cake representation of the Choquet integral.

Proof of Theorem 6.22. We begin with the strong-type estimate (6.18). Let $Q_0 = [0,1)^n$. By Lemma 6.11, there exist constants $C_1, C_2, C_3 > 0$, depending on n and β, such that

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(\mathcal{M}_{\mathcal{H}_\infty^\beta} f\right)^p \, d\mathcal{H}_\infty^\beta &= p\int_0^\infty t^{p-1}\mathcal{H}_\infty^\beta\left(\left\{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^\beta} f(x) > t\right\}\right)\, dt\\
&\leq C_1 p\int_0^\infty t^{p-1}\mathcal{H}_\infty^\beta\left(\left\{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0}} f(x) > C_2 2^{-(\beta+n)}t\right\}\right)\, dt\\
&= C_3\int_{\mathbb{R}^n}\left(\mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0}} f\right)^p \, d\mathcal{H}_\infty^{\beta,Q_0}.
\end{aligned}$$

Moreover, it is easy to see that

$$\mathcal{M}^\#_{\beta,Q_0} f(x) \leq C_\beta \mathcal{M}^\#_\beta f(x),$$

where C_β is the constant in (6.3). Therefore, it suffices to prove that there exists $C' = C'(p,n,\beta)$ such that

$$\int_{\mathbb{R}^n}\left(\mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0}} f\right)^p \, d\mathcal{H}_\infty^{\beta,Q_0} \leq C'\int_{\mathbb{R}^n}\left(\mathcal{M}^\#_{\beta,Q_0} f\right)^p \, d\mathcal{H}_\infty^{\beta,Q_0}. \tag{6.28}$$

To this end, define

$$\phi(t) := \mathcal{H}_\infty^{\beta,Q_0}\left(\left\{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}_\infty^{\beta,Q_0}} f(x) > t\right\}\right).$$

By Theorem 6.24, for every $t > 0$ and every $A > 0$,

$$\phi(t) \leq \mathcal{H}_\infty^{\beta,Q_0}\left(\left\{x\in\mathbb{R}^n : \mathcal{M}^\#_{\beta,Q_0} f(x) > \tfrac{t}{A}\right\}\right) + \frac{8}{A}\,\phi(2^{-\beta-2}t). \tag{6.29}$$

Multiplying by $p\,t^{p-1}$ and integrating over $(0,N)$, we obtain

$$\begin{aligned}
p\int_0^N t^{p-1}\phi(t)\,dt \leq\ & p\int_0^N t^{p-1}\mathcal{H}_\infty^{\beta,Q_0}\left(\left\{x\in\mathbb{R}^n : \mathcal{M}^\#_{\beta,Q_0} f(x) > \tfrac{t}{A}\right\}\right)\, dt\\
&+ \frac{8}{A}\,p\int_0^N t^{p-1}\phi(2^{-\beta-2}t)\,dt.
\end{aligned} \tag{6.30}$$

We estimate the two terms on the right separately. For the first one, the change of variables $u = t/A$ gives

$$p\int_0^N t^{p-1}\mathcal{H}_\infty^{\beta,Q_0}\left(\left\{x\in\mathbb{R}^n : \mathcal{M}^\#_{\beta,Q_0} f(x) > \tfrac{t}{A}\right\}\right)\, dt \leq A^p\int_{\mathbb{R}^n}\left(\mathcal{M}^\#_{\beta,Q_0} f\right)^p \, d\mathcal{H}_\infty^{\beta,Q_0}.$$

For the second one, the change of variables $u = 2^{-\beta-2}t$ yields

$$\frac{8}{A}\, p \int_0^N t^{p-1}\phi(2^{-\beta-2}t)\, dt \le 2^{(\beta+2)p}\frac{8}{A}\, p \int_0^N t^{p-1}\phi(t)\, dt.$$

Substituting these estimates into (6.30), we obtain

$$p\int_0^N t^{p-1}\phi(t)\, dt \le A^p \int_{\mathbb{R}^n} \left(\mathcal{M}^{\#}_{\beta,Q_0} f\right)^p \, d\mathcal{H}^{\beta,Q_0}_\infty + 2^{(\beta+2)p}\frac{8}{A}\, p \int_0^N t^{p-1}\phi(t)\, dt.$$

We now note that the last term on the right-hand side is finite. Indeed, by (6.17), for each $N > 0$,

$$\int_0^N t^{p-1}\phi(t)\, dt \le M_N \int_0^N t^{p-p_0-1}\, dt = M_N \frac{N^{p-p_0}}{p-p_0} < \infty,$$

where

$$M_N := \sup_{0<t<N} t^{p_0}\phi(t) < \infty.$$

Hence the right-hand side is finite, and we may subtract the last term from both sides to obtain

$$\left(1 - 2^{(\beta+2)p}\frac{8}{A}\right) p\int_0^N t^{p-1}\phi(t)\, dt \le A^p \int_{\mathbb{R}^n} \left(\mathcal{M}^{\#}_{\beta,Q_0} f\right)^p \, d\mathcal{H}^{\beta,Q_0}_\infty.$$

We now choose

$$A = 2^{(\beta+2)p+4},$$

so that

$$2^{(\beta+2)p}\frac{8}{A} = \frac{1}{2}.$$

Therefore the previous inequality becomes

$$\frac{1}{2}\, p\int_0^N t^{p-1}\phi(t)\, dt \le A^p \int_{\mathbb{R}^n} \left(\mathcal{M}^{\#}_{\beta,Q_0} f\right)^p \, d\mathcal{H}^{\beta,Q_0}_\infty.$$

Letting $N \to \infty$ and using the monotone convergence theorem, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^n} \left(\mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f\right)^p \, d\mathcal{H}^{\beta,Q_0}_\infty \le 2^{((\beta+2)p+4)p} \int_{\mathbb{R}^n} \left(\mathcal{M}^{\#}_{\beta,Q_0} f\right)^p \, d\mathcal{H}^{\beta,Q_0}_\infty,$$

which proves (6.28), and hence (6.18).

We now turn to the weak-type estimate (6.19). As before, it is enough to show that there exists $C' = C'(p, n, \beta)$ such that

$$\left\|\mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f\right\|_{L^{p,\infty}(\mathcal{H}^{\beta}_\infty)} \le C' \left\|\mathcal{M}^{\#}_{\beta,Q_0} f\right\|_{L^{p,\infty}(\mathcal{H}^{\beta}_\infty)}. \tag{6.31}$$

Fix $N > 0$. Starting again from (6.29), multiply by t^p and take the supremum over $0 < t \le N$. This gives

$$\begin{aligned}
\sup_{0<t\le N} t^p\phi(t) &\le \frac{8}{A}\sup_{0<t\le N} t^p\phi(2^{-\beta-2}t) + \sup_{0<t\le N} t^p \mathcal{H}^{\beta,Q_0}_\infty\left(\left\{x\in\mathbb{R}^n : \mathcal{M}^{\#}_{\beta,Q_0}f(x) > \tfrac{t}{A}\right\}\right)\\
&= \frac{2^{(\beta+2)p+3}}{A}\sup_{0<t\le 2^{-\beta-2}N} t^p\phi(t) + A^p \sup_{0<t\le N/A} t^p \mathcal{H}^{\beta,Q_0}_\infty\left(\left\{x\in\mathbb{R}^n : \mathcal{M}^{\#}_{\beta,Q_0}f(x) > t\right\}\right)\\
&\le \frac{2^{(\beta+2)p+3}}{A}\sup_{0<t\le N} t^p\phi(t) + A^p \left\|\mathcal{M}^{\#}_{\beta,Q_0} f\right\|^p_{L^{p,\infty}(\mathcal{H}^{\beta}_\infty)}.
\end{aligned}$$

Again, the left-hand side is finite because

$$\sup_{0<t\leq N} t^p \phi(t) \leq M_N N^{p-p_0} < \infty.$$

Choosing once more

$$A = 2^{(\beta+2)p+4},$$

we get

$$\frac{2^{(\beta+2)p+3}}{A} = \frac{1}{2}.$$

Hence we may absorb the first term on the right-hand side and conclude that

$$\sup_{0<t\leq N} t^p \phi(t) \leq 2^{((\beta+2)p+4)p+1} \left\| \mathcal{M}^{\#}_{\beta,Q_0} f \right\|^p_{L^{p,\infty}(\mathcal{H}^\beta_\infty)}.$$

Letting $N \to \infty$ yields (6.31). This completes the proof. □

6.4 Riesz potentials and Hausdorff content

A classical theorem of Adams identifies the oscillation of the Riesz potential through the sharp maximal function. In **[Ada75]**, Adams proved the pointwise equivalence between the classical sharp maximal function of $I_\alpha f$ and the fractional maximal function $\mathcal{M}_\alpha f$. Our second main result is a refinement of this fact in the context of the β-dimensional sharp maximal operator. We shall also collect several capacitary estimates for Riesz potentials and fractional maximal operators that will be used in later sections.

6.4.1 Pointwise estimates for Riesz potentials

An approach to proving the Muckenhoupt–Wheeden inequality, stated in Theorem 1.26, relies on an interesting pointwise estimate due to Adams **[Ada75]**.

Theorem 6.25. *Let $n \in \mathbb{N}$ and $0 < \alpha < n$. The pointwise inequality*

$$\mathcal{M}^{\#}\big(I_\alpha f\big)(x) \;\simeq\; \mathcal{M}_\alpha f(x), \tag{6.32}$$

holds for all non-negative measurable functions f in $\mathbb{R}^n$ such that $I_\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a tempered distribution.

Remark 6.26. In **[Ada75]**, the hypothesis given for (6.32) requires $I_\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ along with the integrability condition

$$\int_{\mathbb{R}^n} \frac{I_\alpha f(x)}{(1+|x|)^{n+\alpha}}\, dx < \infty.$$

However, a closer examination of **[Ada75, p. 771]** shows that this integrability condition is not strictly necessary. In fact, the weaker assumption that $I_\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and is a tempered distribution suffices for the pointwise estimate (6.32).

Our refinement of (6.32) is the following.

Theorem 6.27. *Let $n \in \mathbb{N}$ and $0 < \alpha < n$, $\beta \in (n-\alpha, n]$. The pointwise inequality*

$$\mathcal{M}^{\#}_{\beta}(I_\alpha f)(x) \simeq \mathcal{M}_\alpha f(x)$$

holds for all non-negative measurable functions f in $\mathbb{R}^n$ such that $I_\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a tempered distribution.

Theorem 6.27 implies an analogue of **[Che25, Theorem 1.6]** in terms of Morrey spaces. More precisely, if $f \geq 0$ is such that $I_\alpha f$ is a tempered distribution, then

$$\|I_\alpha f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \simeq \|f\|_{\mathscr{M}^{n-\alpha}(\mathbb{R}^n)},$$

for every $\beta \in (n-\alpha, n]$. This may be interpreted as a self-improving phenomenon. Specifically, when $\beta < n$, the space BMO$^\beta$ is strictly smaller than the classical space BMO, and by combining Theorem 6.27 with Adams's pointwise estimate (6.32), we obtain that for every $\beta \in (n-\alpha, n]$,

$$\|I_\alpha f\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \simeq \|I_\alpha f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

To prove Theorem 6.27, we first collect several auxiliary results. We begin with the following estimate, which is taken from **[COS24, Lemma 2.2]** and **[HH24, Proposition 2.3]**, and extends **[OV98, Lemma 3]**.

Lemma 6.28. *Let $n \in \mathbb{N}$ with $0 < \alpha \leq \beta \leq n$ and let f be a nonnegative function. Then*

$$\int_{\mathbb{R}^n} f \, d\mathcal{H}^\beta_\infty \leq \frac{\beta}{\alpha} \left(\int_{\mathbb{R}^n} f^{\frac{\alpha}{\beta}} \, d\mathcal{H}^\alpha_\infty \right)^{\frac{\beta}{\alpha}}.$$

As an immediate consequence, we obtain the following comparison between sharp maximal operators of different dimensions.

Lemma 6.29. *Let $n \in \mathbb{N}$ with $0 < \alpha \leq \beta \leq n$. Then there exists a constant $C > 0$ depending only on α and β such that*

$$\mathcal{M}^{\#}_\beta f(x) \leq C \mathcal{M}^{\#}_\alpha f(x),$$

for all $x \in \mathbb{R}^n$.

Proof. Let $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq n$. Fix $x \in \mathbb{R}^n$ and let Q be a cube containing x. Define c_Q by

$$c_Q := \operatorname*{arg\,min}_{c \in \mathbb{R}} \frac{1}{\ell(Q)^\alpha} \int_Q |f - c_Q| \, d\mathcal{H}^\alpha_\infty.$$

An application of Lemma 6.28 and Hölder's inequality in Lemma 6.5 (iii) yields that

$$\begin{aligned}
\frac{1}{\mathcal{H}^\beta_\infty(Q)} \int_Q |f - c_Q| \, d\mathcal{H}^\beta_\infty &\leq \frac{\beta}{\alpha} \frac{1}{\mathcal{H}^\beta_\infty(Q)} \left(\int_Q |f - c_Q|^{\frac{\alpha}{\beta}} \, d\mathcal{H}^\alpha_\infty \right)^{\frac{\beta}{\alpha}} \\
&\leq \frac{2\beta \mathcal{H}^\alpha_\infty(Q)^{\frac{\beta}{\alpha}}}{\alpha \mathcal{H}^\beta_\infty(Q)} \left(\frac{1}{\mathcal{H}^\alpha_\infty(Q)} \int_Q |f - c_Q| \, d\mathcal{H}^\alpha_\infty \right) \\
&\leq C \mathcal{M}^{\#}_\alpha f(x),
\end{aligned}$$

where the constant

$$C = \frac{2\beta \mathcal{H}^{\alpha}_{\infty}(Q)^{\frac{\beta}{\alpha}}}{\alpha \mathcal{H}^{\beta}_{\infty}(Q)}$$

does not depend on Q because $\mathcal{H}^{\gamma}_{\infty}(Q) \simeq \ell(Q)^{\gamma}$ for any $\gamma > 0$, implying $\mathcal{H}^{\alpha}_{\infty}(Q)^{\beta/\alpha} \simeq \ell(Q)^{\beta} \simeq \mathcal{H}^{\beta}_{\infty}(Q)$. Taking the supremum over all cubes Q such that $x \in Q$, we obtain

$$\mathcal{M}^{\#}_{\beta} f(x) \leq C \mathcal{M}^{\#}_{\alpha} f(x).$$

□

We next record two local estimates from **[Che25, Lemma 3.5 and Lemma 3.8]**, which will be the basic ingredients in the proof of the sharp maximal estimate.

Lemma 6.30. *Let $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq n$, $\beta \in (n-\alpha, n]$. Let Q be an open cube in $\mathbb{R}^n$ with center x_0 and f be a real-valued measurable function in $\mathbb{R}^n$. If $\operatorname{supp} f \subseteq 2Q$, then $I_\alpha f \in L^1(Q; \mathcal{H}^{\beta}_{\infty})$ and there exists a constant C depending only on n, α and β such that*

$$\int_Q |I_\alpha f| \, d\mathcal{H}^{\beta}_{\infty} \leq C \ell(Q)^{\beta} \mathcal{M}_{\alpha} f(x_0). \tag{6.33}$$

Lemma 6.31. *Let $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq n$, $\beta \in (0, n]$. Let Q be an open cube in $\mathbb{R}^n$ with center x_0 and f be a real-valued measurable function with $\operatorname{supp} f \subseteq (2Q)^c$. If $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$, then $I_\alpha f$ is continuous in Q and there exists $c \in \mathbb{R}$ such that*

$$\int_Q |I_\alpha f - c| \, d\mathcal{H}^{\beta}_{\infty} \leq C \ell(Q)^{\beta} \mathcal{M}_{\alpha} f(x_0). \tag{6.34}$$

For the proof of the next result, we introduce the β-dimensional centered sharp maximal operator $\mathcal{M}^{\#,c}_{\beta} f(x)$ of a function f, which is defined as

$$\mathcal{M}^{\#,c}_{\beta} f(x) := \sup_{r>0} \inf_{c \in \mathbb{R}} \frac{1}{r^{\beta}} \int_{Q(x,r)} |f - c| \, d\mathcal{H}^{\beta}_{\infty}.$$

It is easy to see that there exists a constant $C_\beta > 0$ such that

$$\mathcal{M}^{\#,c}_{\beta} f(x) \leq \mathcal{M}^{\#}_{\beta} f(x) \leq C_\beta \mathcal{M}^{\#,c}_{\beta} f(x). \tag{6.35}$$

The previous local lemmas readily imply the following pointwise bound.

Theorem 6.32. *Let $n \in \mathbb{N}$, $0 < \alpha < n$ and $\beta \in (n-\alpha, n]$. If $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$, then for every $x \in \mathbb{R}^n$, there exists a constant C depending only on n, α, β such that*

$$\mathcal{M}^{\#}_{\beta}(I_\alpha f)(x) \leq C \mathcal{M}_{\alpha} f(x). \tag{6.36}$$

Proof of Theorem 6.32. Suppose that $I_\alpha f \in L^1_{loc}(\mathbb{R}^n)$. Let Q be a cube in $\mathbb{R}^n$ with center x and $f_1 = f\chi_{2Q}$, $f_2 = f - f_1$. Then there exists a constant $c \in \mathbb{R}$ and C depending on n, α, β such that

$$\begin{aligned}
\int_Q |I_\alpha f - c| \, d\mathcal{H}^{\beta}_{\infty} &= \int_Q |I_\alpha f_1 + I_\alpha f_2 - c| \, d\mathcal{H}^{\beta}_{\infty} \\
&\leq 2 \left(\int_Q |I_\alpha f_1| \, d\mathcal{H}^{\beta}_{\infty} + \int_Q |I_\alpha f_2 - c| \, d\mathcal{H}^{\beta}_{\infty} \right) \\
&\leq 2C \ell(Q)^{\beta} \mathcal{M}_{\alpha} f(x),
\end{aligned}$$

where we use Lemma 6.30 and Lemma 6.31 in the last inequality. Now dividing both sides with $\ell(Q)^\beta$ and taking the supremum over all cubes Q with center x, we obtain

$$\mathcal{M}_\beta^{\#,c}(I_\alpha f)(x) \leq C\mathcal{M}_\alpha f(x)$$

and thus

$$\mathcal{M}_\beta^{\#}(I_\alpha f)(x) \leq C\mathcal{M}_\alpha f(x)$$

by (6.35). □

We are now in a position to prove the refinement of Adams's theorem.

Proof of Theorem 6.27. Let $x \in \mathbb{R}^n$. By Theorem 6.32, we have

$$\mathcal{M}_\beta^{\#}(I_\alpha f)(x) \lesssim \mathcal{M}_\alpha f(x).$$

The combination of (6.32) and Lemma 6.29 yields that

$$\mathcal{M}_\alpha f(x) \simeq \mathcal{M}^{\#}(I_\alpha f)(x) \lesssim \mathcal{M}_\beta^{\#}(I_\alpha f)(x),$$

where we note that $\mathcal{M}_n^{\#} \simeq \mathcal{M}^{\#}$ (see for instance **[HH26b, Remark 3.4]**). This completes the proof. □

6.4.2 Further capacitary estimates

We conclude the section by collecting several capacitary estimates for I_α and the fractional maximal operator that will be used later. The first result is the boundedness of the Riesz potential I_α in Lebesgue spaces with respect to the Hausdorff content $\mathcal{H}_\infty^\beta$, established in **[HH26b, Theorem 5.4]**.

Theorem 6.33. *Let $n \in \mathbb{N}$, $0 < \beta \leq n$, $\alpha \in (0, \beta)$, and $p \in (\frac{\beta}{n}, \frac{\beta}{\alpha})$ be given. Then, for all measurable functions $f : \mathbb{R}^n \longrightarrow [-\infty, \infty]$, we have*

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^{\frac{\beta p}{\beta - p\alpha}} \, d\mathcal{H}_\infty^\beta\right)^{\frac{\beta - p\alpha}{\beta p}} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p \, d\mathcal{H}_\infty^\beta\right)^{\frac{1}{p}},$$

where $C = C(n, \beta, \alpha, p) > 0$.

We will also require the boundedness of the fractional maximal operator $\mathcal{M}_{\alpha, \mathcal{H}_\infty^\beta}$, defined by

$$\mathcal{M}_{\alpha, \mathcal{H}_\infty^\beta} f(x) = \sup_{x \in Q} \frac{\ell(Q)^\alpha}{\mathcal{H}_\infty^\beta(Q)} \int_Q |f| \, d\mathcal{H}_\infty^\beta.$$

This follows readily from **[HH26b, Theorem 5.11]** via the pointwise estimate $\mathcal{M}_{\alpha, \mathcal{H}_\infty^\beta} f(x) \lesssim \mathcal{R}_\alpha^\beta f(x)$, where $\mathcal{R}_\alpha^\beta f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mathcal{H}_\infty^\beta$ denotes the β-dimensional Riesz potential of order α.

Theorem 6.34. *Let $n \in \mathbb{N}$, $0 < \beta \leq n$, $\alpha \in (0, \beta)$, and $p \in (1, \frac{\beta}{\alpha})$ be given. Then, there exists a constant $C = C(n, \beta, \alpha, p) > 0$ such that*

$$\left(\int_{\mathbb{R}^n} \mathcal{M}_{\alpha, \mathcal{H}_\infty^\beta} f(x)^{\frac{\beta p}{\beta - p\alpha}} \, d\mathcal{H}_\infty^\beta\right)^{\frac{\beta - p\alpha}{\beta p}} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p \, d\mathcal{H}_\infty^\beta\right)^{\frac{1}{p}},$$

for every function $f : \mathbb{R}^n \longrightarrow [-\infty, \infty]$.

The next two lemmas are slight modifications of **[Che25, Lemma 3.5 and Lemma 3.8]** that will be used later in the capacitary setting.

Lemma 6.35. *Let $n \in \mathbb{N}$, $0 < \alpha < \beta \le n$. Let Q be an open cube in $\mathbb{R}^n$ with center x_0, and let f be a measurable function defined on $\mathbb{R}^n$. If* $\operatorname{supp}(f) \subseteq 2Q$, *then there exists a constant $C = C(n, \alpha, \beta) > 0$ such that*

$$\int_Q |I_\alpha f| \, d\mathcal{H}^\beta_\infty \le C\ell(Q)^\alpha \int_{2Q} |f| \, d\mathcal{H}^\beta_\infty.$$

Proof. First, recall that for any $\eta \in [0, \alpha)$, there exists a constant $C_\eta > 0$ such that

$$|I_\alpha \phi(x)| \le C_\eta \ell(Q)^{\alpha-\eta} \mathcal{M}_\eta \phi(x), \tag{6.37}$$

which can be found, for instance, in **[Che25, (3.6)]**. Additionally, by **[HH26b, Theorem 5.4]**, there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} (\mathcal{M}_\eta f)^{\frac{\beta}{\beta-\eta}} \, d\mathcal{H}^\beta_\infty \right)^{\frac{\beta-\eta}{\beta}} \le C \int_{\mathbb{R}^n} |f| \, d\mathcal{H}^\beta_\infty. \tag{6.38}$$

The combination of (6.37) and (6.38) then yields that

$$\begin{aligned}
\int_Q |I_\alpha f| \, d\mathcal{H}^\beta_\infty &\le C\ell(Q)^{\alpha-\eta} \int_Q \mathcal{M}_\eta f \, d\mathcal{H}^\beta_\infty \\
&\le C\ell(Q)^{\alpha-\eta} \left(\int_Q (\mathcal{M}_\eta f)^{\frac{\beta}{\beta-\eta}} \, d\mathcal{H}^\beta_\infty \right)^{\frac{\beta-\eta}{\beta}} \left(\mathcal{H}^\beta_\infty(Q) \right)^{\frac{\eta}{\beta}} \\
&\le C\ell(Q)^\alpha \int_{2Q} |f| \, d\mathcal{H}^\beta_\infty.
\end{aligned}$$

This completes the proof. □

Lemma 6.36. *Let $n \in \mathbb{N}$, $0 < \alpha < n$, and $\beta \in (0, n]$. Let Q be an open cube in $\mathbb{R}^n$ with center x_0 and f be a measurable function with* $\operatorname{supp} f \subseteq (2Q)^c$. *Then, there exists $c \in \mathbb{R}$, and $C = C(n, \beta, \alpha) > 0$ such that*

$$\frac{1}{\ell(Q)^\beta} \int_Q |I_\alpha f - c| \, d\mathcal{H}^\beta_\infty \le C \sum_{k=0}^\infty 2^{-k} (2^{k+1}\ell(Q))^\alpha \frac{1}{(2^{k+1}\ell(Q))^\beta} \int_{2^{k+1}Q} |f| d\mathcal{H}^\beta_\infty.$$

Proof. It is enough to prove the result for a cube Q with center $x_0 = 0$. We follow the proof of **[Che25, Lemma 3.8]**, taking $c = I_\alpha f(0)$ until (3.24),

$$\begin{aligned}
\frac{1}{\ell(Q)^\beta} \int_Q |I_\alpha f - c| d\mathcal{H}^\beta_\infty \le& \frac{C}{\ell(Q)^\beta} \int_Q \int_{(2Q)^c} \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right| |f(y)| dy d\mathcal{H}^\beta_\infty(x) \\
\le& \frac{C}{\ell(Q)^\beta} \int_Q |x| d\mathcal{H}^\beta_\infty(x) \int_{(2Q)^c} \frac{|f(y)|}{|y|^{n-\alpha+1}} dy \\
\le& \frac{C}{\ell(Q)^\beta} \ell(Q)^{\beta+1} \int_{|y| \ge \ell(Q)} \frac{|f(y)|}{|y|^{n-\alpha+1}} dy \\
=& C\ell(Q) \sum_{k=0}^\infty \int_{2^{k-1}\ell(Q) \le |y| < 2^k \ell(Q)} \frac{|f(y)|}{|y|^{n-\alpha+1}} dy
\end{aligned}$$

$$\begin{aligned}
&\leq C\ell(Q)\sum_{k=0}^{\infty}(2^{k-1}\ell(Q))^{-n+\alpha-1}\int_{|y|<2^k\ell(Q)}|f(y)|dy\\
&\leq C\sum_{k=0}^{\infty}2^{-k}(2^{k-1}\ell(Q))^{-n+\alpha}\int_{2^{k+1}Q}|f(y)|dy\\
&= C'\sum_{k=0}^{\infty}2^{-k}(2^{k+1}\ell(Q))^{-n+\alpha}\int_{2^{k+1}Q}|f(y)|dy
\end{aligned}$$

If $\beta = n$, we have concluded the proof. Let us assume $\beta < n$, then using Lemma 6.28, and Hölder inequality (iii) of Lemma 6.5, we have

$$\begin{aligned}
\int_{2^{k+1}Q}|f(y)|dy &\leq \frac{n}{\beta}\left(\int_{2^{k+1}Q}|f|^{\frac{\beta}{n}}d\mathcal{H}^\beta_\infty\right)^{\frac{n}{\beta}}\\
&\leq 2\frac{n}{\beta}\left(\int_{2^{k+1}Q}|f|d\mathcal{H}^\beta_\infty\right)\ell(2^{k+1}Q)^{\beta\frac{\frac{n}{\beta}}{(\frac{n}{\beta})'}}\\
&= 2\frac{n}{\beta}(2^{k+1}\ell(Q))^{n-\beta}\int_{2^{k+1}Q}|f|d\mathcal{H}^\beta_\infty.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_Q|I_\alpha f - c|d\mathcal{H}^\beta_\infty &\leq C\sum_{k=0}^{\infty}2^{-k}(2^{k+1}\ell(Q))^{-n+\alpha}\int_{2^{k+1}Q}|f(y)|dy\\
&\leq C'\sum_{k=0}^{\infty}2^{-k}(2^{k+1}\ell(Q))^{-n+\alpha}(2^{k+1}\ell(Q))^{n-\beta}\int_{2^{k+1}Q}|f|d\mathcal{H}^\beta_\infty\\
&= C'\sum_{k=0}^{\infty}2^{-k}(2^{k+1}\ell(Q))^{\alpha}\frac{1}{(2^{k+1}\ell(Q))^\beta}\int_{2^{k+1}Q}|f|d\mathcal{H}^\beta_\infty.
\end{aligned}$$

□

We next prove that Riesz potentials satisfy a capacitary A_1-type property.

Lemma 6.37. *Let $0 < \alpha < n$ and $0 < \beta \leq n$. Then*

$$\mathcal{M}_{\mathcal{H}^\beta_\infty}(I_\alpha f)(x) \leq (\mathcal{M}_{\mathcal{H}^\beta_\infty}I_\alpha) * |f|(x),$$

for any $x \in \mathbb{R}^n$, where $I_\alpha(x) = |x|^{\alpha-n}$. In addition, if $\beta \in (n-\alpha, n]$, we have

$$\mathcal{M}_{\mathcal{H}^\beta_\infty}(I_\alpha f)(x) \leq C\, I_\alpha(|f|)(x)$$

for $\mathcal{H}^\beta_\infty$-a.e. $x \in \mathbb{R}^n$. That is, if $\beta \in (n-\alpha, n]$ and f is a nonnegative function, then $I_\alpha f \in A^\beta_1$.

Proof. We may assume without loss of generality that f is nonnegative. Fix $z \in \mathbb{R}^n$ and a cube Q containing z. Since $I_\alpha f$ is lower semicontinuous, we may apply **[Ada88, Corollary on p. 118]** and we obtain

$$\begin{aligned}\frac{1}{\mathcal{H}^\beta_\infty(Q)}\int_Q I_\alpha f(x)\,d\mathcal{H}^\beta_\infty &= \sup_{\|\mu\|_{\mathscr{M}^\beta(\mathbb{R}^n)}\le 1}\frac{1}{\mathcal{H}^\beta_\infty(Q)}\int_Q I_\alpha f(x)\,d\mu(x)\\ &= \sup_{\|\mu\|_{\mathscr{M}^\beta(\mathbb{R}^n)}\le 1}\frac{1}{\mathcal{H}^\beta_\infty(Q)}\int_Q\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy\,d\mu(x)\\ &= \sup_{\|\mu\|_{\mathscr{M}^\beta(\mathbb{R}^n)}\le 1}\int_{\mathbb{R}^n} f(y)\left(\frac{1}{\mathcal{H}^\beta_\infty(Q)}\int_Q\frac{d\mu(x)}{|x-y|^{n-\alpha}}\right)dy, \end{aligned}\tag{6.39}$$

where we have used Fubini's theorem. As $\mu(E)\le\mathcal{H}^\beta_\infty(E)$ for any measurable set $E\subseteq\mathbb{R}^n$, we have

$$\begin{aligned}\frac{1}{\mathcal{H}^\beta_\infty(Q)}\int_Q\frac{d\mu(x)}{|x-y|^{n-\alpha}} &\le \frac{1}{\mathcal{H}^\beta_\infty(Q)}\int_Q\frac{1}{|x-y|^{n-\alpha}}\,d\mathcal{H}^\beta_\infty(x)\\ &\le \mathcal{M}_{\mathcal{H}^\beta_\infty}(I_\alpha)(z-y).\end{aligned}\tag{6.40}$$

The combination of (6.39) and (6.40) then yields that

$$\begin{aligned}\frac{1}{\mathcal{H}^\beta_\infty(Q)}\int_Q I_\alpha f(x)\,d\mathcal{H}^\beta_\infty(x) &\le \int_{\mathbb{R}^n} f(y)\,\mathcal{M}_{\mathcal{H}^\beta_\infty}(I_\alpha)(z-y)\,dy\\ &= (\mathcal{M}_{\mathcal{H}^\beta_\infty} I_\alpha) * f(z).\end{aligned}$$

Taking the supremum over all such cubes Q, we obtain

$$\mathcal{M}_{\mathcal{H}^\beta_\infty}(I_\alpha f)(z)\le(\mathcal{M}_{\mathcal{H}^\beta_\infty} I_\alpha)*f(z).$$

To conclude the proof, we use the fact (see **[HZZ25, Proposition 2.14]**) that if $\beta\in(n-\alpha,n]$, then I_α is an A_1^β weight, that is,

$$\mathcal{M}_{\mathcal{H}^\beta_\infty} I_\alpha(x)\le C\,I_\alpha(x)$$

for $\mathcal{H}^\beta_\infty$-a.e. $x\in\mathbb{R}^n$. This, together with the previous estimate, completes the proof. □

We now state the reverse Hölder inequality from **[HZZ25, Theorem 1.7]** in the particular case of A_1^β weights, which will be needed in what follows.

Theorem 6.38. *Let $0<\beta\le n$ and $w\in A_1^\beta$ (that is, $\mathcal{M}_{\mathcal{H}^\beta_\infty} w(x)\le Cw(x)$ for $\mathcal{H}^\beta_\infty$-a.e. x). There exist $C>0$ and $r>0$ such that*

$$\left(\frac{1}{\ell(Q)^\beta}\int_Q w^{1+r}\,d\mathcal{H}^\beta_\infty\right)^{\frac{1}{1+r}}\le C\frac{1}{\ell(Q)^\beta}\int_Q w\,d\mathcal{H}^\beta_\infty,$$

for every cube Q.

6.5 Capacitary Muckenhoupt–Wheeden inequality

Let $n \in \mathbb{N}$ and $0 < \alpha < n$. We recall that the Riesz potential of order α of a locally finite measure μ is defined by

$$I_\alpha \mu(x) := \gamma_{n,\alpha} \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}},$$

where $\gamma_{n,\alpha}$ is given by (5.2). The corresponding fractional maximal function $\mathcal{M}_\alpha \mu$ is defined as

$$\mathcal{M}_\alpha \mu(x) := \sup_{Q \ni x} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}},$$

where the supremum is taken over all cubes Q containing x.

It is easy to see that $\mathcal{M}_\alpha \mu(x) \lesssim_{n,\alpha} I_\alpha \mu(x)$ for every $x \in \mathbb{R}^n$. Conversely, the classical Muckenhoupt–Wheeden inequality (Theorem 1.26) shows that the $L^p(\mathbb{R}^n)$ norms of these two operators are in fact comparable. The standard proof of Theorem 1.26 relies heavily on a good-lambda inequality. Theorems 6.22 and 6.27 naturally raise the question of whether an analogue of the Muckenhoupt–Wheeden inequality holds in the context of the capacitary Lebesgue spaces $L^p(\mathcal{H}^\beta_\infty)$. We answer this question in the affirmative, thereby extending the classical estimate to this nonlinear setting. Note that the standard Muckenhoupt–Wheeden inequality is recovered as a special case when $\beta = n$.

Theorem 6.39. *Let $0 < \alpha < n \in \mathbb{N}$, $\beta \in (n-\alpha, n]$, $1 \le p < \infty$ and μ be a locally finite measure in $\mathbb{R}^n$. Then there exists $C = C(n,\alpha,\beta) > 0$ such that*

$$\left(\int_{\mathbb{R}^n} (I_\alpha \mu)^p \ d\mathcal{H}^\beta_\infty \right)^{\frac{1}{p}} \le C\, p \left(\int_{\mathbb{R}^n} (\mathcal{M}_\alpha \mu)^p \ d\mathcal{H}^\beta_\infty \right)^{\frac{1}{p}}, \tag{6.41}$$

and

$$\|I_\alpha \mu\|_{L^{p,\infty}(\mathcal{H}^\beta_\infty)} \le C\, p \|\mathcal{M}_\alpha \mu\|_{L^{p,\infty}(\mathcal{H}^\beta_\infty)}. \tag{6.42}$$

It is noteworthy that the proof of Theorem 6.39 does not rely on Theorem 6.27. In fact, if one were to use Theorem 6.27 to derive Theorem 6.39, additional assumptions on $I_\alpha f$ would be required. Instead, our approach employs the good-lambda estimate established in Theorem 6.44.

Remark 6.40. The linear dependence on p is a consequence of the exponential decay in our good-lambda inequality, stated in Theorem 6.45.

Remark 6.41. The dimensional constraint $\beta \in (n-\alpha, n]$ in Theorems 6.27 and 6.39 naturally arises from the interaction between the Riesz potential I_α and the Hausdorff content $\mathcal{H}^\beta_\infty$. When $\beta \le n-\alpha$, the convolution kernel $|x|^{\alpha-n}$ fails to be integrable near the origin with respect to the content $\mathcal{H}^\beta_\infty$. This non-integrability implies that $I_\alpha f$ may be infinite on sets of dimension $n-\alpha$ for general functions f, as observed by Adams **[Ada75, p. 772]**. On the other hand, under additional assumptions as $f \in L^r(\mathbb{R}^n)$ for some $r \in (1, \frac{n}{\alpha})$, then $I_\alpha f$ is in $L^p(\mathcal{H}^{n-\alpha}_\infty)$ with $p = \frac{r(n-\alpha)}{n-\alpha r}$, as established in **[HH26b, Theorem 5.9]**. However, for general functions f, the constraint $\beta \in (n-\alpha, n]$ is necessary.

In order to prove Theorem 6.39, we first prove a good-lambda inequality with exponential decay, following the ideas of **[HJ12]** and using the packing condition to avoid difficulties due to the nonlinearity of the Hausdorff content.

First, we define a dyadic version of the Riesz potential, which we will use to approximate I_α. We begin with the definition of a dyadic grid. A dyadic grid $\mathcal{D}$ is a countable family of cubes with the following properties:

1. For each $Q \in \mathcal{D}$, $\ell(Q) = 2^k$ for some $k \in \mathbb{Z}$.
2. Given $Q, P \in \mathcal{D}$, then $Q \cap P \in \{\emptyset, P, Q\}$.
3. For each $k \in \mathbb{Z}$, the set $\mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^k\}$ is a partition of $\mathbb{R}^n$.

The following dyadic operator will play a central role in the proof of the good-lambda estimate.

Definition 6.42. Let $0 < \alpha < n \in \mathbb{N}$ and let $\mathcal{D}$ be a dyadic grid. We define the dyadic Riesz potential with respect to $\mathcal{D}$ of a locally finite measure μ, by

$$I_\alpha^{\mathcal{D}}\mu(x) = \sum_{Q \in \mathcal{D}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}} \chi_Q(x).$$

Dyadic Riesz potentials were first introduced by Sawyer and Wheeden **[SW92]**. We will use the following result by Cruz-Uribe and Moen (see **[CM13, Proposition 2.2]**) to approximate the Riesz potential by a linear combination of dyadic Riesz potentials.

Proposition 6.43. *Let $n \in \mathbb{N}$, $0 < \alpha < n$. Let μ be a locally finite measure, then for any dyadic grid $\mathcal{D}$,*

$$I_\alpha^{\mathcal{D}}\mu(x) \leq c_{n,\alpha} I_\alpha \mu(x). \tag{6.43}$$

Conversely, we have that

$$I_\alpha \mu(x) \leq c_{n,\alpha} \max_{i \in \{0, \frac{1}{3}\}^n} I_\alpha^{\mathcal{D}^i}\mu(x),$$

where

$$\mathcal{D}^i = \{2^{-k}([0,1)^n + m + (-1)^k i) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \qquad i \in \left\{0, \frac{1}{3}\right\}^n.$$

Hence, the Riesz potential is pointwise equivalent to a linear combination of dyadic Riesz potentials, that is,

$$I_\alpha \mu(x) \simeq_{n,\alpha} \sum_{i \in \{0, \frac{1}{3}\}^n} I_\alpha^{\mathcal{D}^i}\mu(x). \tag{6.44}$$

We should mention that the proof of the previous result in **[CM13]** is made for functions, but a detailed inspection of the proof shows that the same proof works for measures.

Now, we state our good-lambda estimate. The result is an extension of the good-lambda inequality proved in **[HJ12]**, which improves the classical result of Muckenhoupt and Wheeden **[MW74]**.

Theorem 6.44. *Let $0 < \alpha < n$, $\beta \in (n - \alpha, n]$ and let $\mathcal{D}$ be a dyadic grid. Then, there exist constants $C, c > 0$ depending on n, α, β such that*

$$\mathcal{H}_\infty^\beta(\{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda, \mathcal{M}_\alpha \mu(x) \leq \varepsilon\lambda\}) \leq C e^{-\frac{c}{\varepsilon}} \mathcal{H}_\infty^\beta(\{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > \lambda\}), \tag{6.45}$$

for all $\lambda > 0$ and all $0 < \varepsilon < 1$ and all locally finite measures μ.

Proof. It is sufficient to prove (6.45) for the dyadic content $\mathcal{H}_\infty^{\beta,Q_0}$, where Q_0 is a cube in $\mathcal{D}$, since both contents are equivalent by (6.4).

Fix $\lambda > 0$ and $0 < \varepsilon < 1$. We denote $G_\lambda = \{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > \lambda\}$. We can assume without loss of generality $\mathcal{H}_\infty^{\beta,Q_0}(G_\lambda) < \infty$. Let $\{Q_j^\lambda\}_j$ be the maximal collection of dyadic cubes subordinate to $\mathcal{D}$ contained in G_λ with respect to the inclusion. Applying Lemma 6.7 to the family $\{Q_j^\lambda\}_j$ we obtain a subfamily $\{Q_{j_k}^\lambda\}_k$ and a family of non-overlapping ancestors $\{\tilde{Q}_k\}_k$ such that

1.
$$G_\lambda = \bigcup_j Q_j^\lambda \subset \bigcup_k Q_{j_k}^\lambda \cup \bigcup_k \tilde{Q}_k$$

2.
$$\sum_{Q_{j_k}^\lambda \subset Q} \ell(Q_{j_k}^\lambda)^\beta \leq 2\ell(Q)^\beta, \text{ for each dyadic cube } Q.$$

3.
$$\begin{aligned}\mathcal{H}_\infty^{\beta,Q_0}(\cup_j Q_j^\lambda) &\leq \sum_{k, Q_{j_k}^\lambda \not\subseteq \tilde{Q}_m} \ell(Q_{j_k}^\lambda)^\beta + \sum_k \ell(\widetilde{Q}_k)^\beta \\ &\leq 2\sum_k \ell(Q_{j_k}^\lambda)^\beta \\ &\leq 2\mathcal{H}_\infty^{\beta,Q_0}(\cup_j Q_j^\lambda).\end{aligned}$$

Define $S = \{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda, \mathcal{M}_\alpha\mu(x) \leq \varepsilon\lambda\}$. Since $G_{2\lambda} \subseteq G_\lambda$, we observe that

$$S \subseteq G_{2\lambda} \subseteq G_\lambda = \bigcup_k Q_{j_k}^\lambda \cup \bigcup_k \tilde{Q}_k,$$

and then we have

$$\begin{aligned}S \subseteq &\bigcup_{k: Q_{j_k}^\lambda \not\subseteq \tilde{Q}_m} \{x \in Q_{j_k}^\lambda : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda, \mathcal{M}_\alpha\mu(x) \leq \varepsilon\lambda\} \\ &\cup \bigcup_k \{x \in \tilde{Q}_k : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda, \mathcal{M}_\alpha\mu(x) \leq \varepsilon\lambda\}.\end{aligned}$$

Using the subadditivity of $\mathcal{H}_\infty^{\beta,Q_0}$ we obtain

$$\begin{aligned}\mathcal{H}_\infty^{\beta,Q_0}(S) \leq &\sum_{k: Q_{j_k}^\lambda \not\subseteq \tilde{Q}_m} \mathcal{H}_\infty^{\beta,Q_0}(\{x \in Q_{j_k}^\lambda : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda, \mathcal{M}_\alpha\mu(x) \leq \varepsilon\lambda\}) \\ &+ \sum_k \mathcal{H}_\infty^{\beta,Q_0}(\{x \in \tilde{Q}_k : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda, \mathcal{M}_\alpha\mu(x) \leq \varepsilon\lambda\}).\end{aligned} \tag{6.46}$$

Fix $Q' \in \{Q_{j_k}^\lambda\}_k \cup \{\tilde{Q}_k\}_k$, and let us define $S_{Q'} = \{x \in Q' : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda, \mathcal{M}_\alpha\mu(x) \leq \varepsilon\lambda\}$. We can assume that $S_{Q'}$ is non-empty. We have the following estimate on the tail of the potential

$$\sum_{\substack{Q \in \mathcal{D} \\ Q \supsetneq Q'}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}} \leq \lambda,$$

by the definition of G_λ and observing that either Q' is a maximal cube of the family $\{Q_j^\lambda\}_j$, or it contains a maximal cube in $\{Q_j^\lambda\}_j$. Let $j_0 \in \mathbb{Z}$ such that $\ell(Q') = 2^{j_0}$ and $m \in \mathbb{N}$ to be chosen later. If $x \in S_{Q'}$, we have

$$\begin{aligned} I_\alpha^{\mathcal{D}}\mu(x) &= \sum_{x\in Q\in\mathcal{D}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}} \\ &= \sum_{\substack{x\in Q\in\mathcal{D}\\ Q\supsetneq Q'}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}} + \sum_{\substack{x\in Q\in\mathcal{D}\\ 2^{j_0-m}<\ell(Q)\le 2^{j_0}}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}} + \sum_{\substack{x\in Q\in\mathcal{D}\\ \ell(Q)\le 2^{j_0-m}}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}} \\ &\le \lambda + m\mathcal{M}_\alpha\mu(x) + \sum_{\substack{x\in Q\in\mathcal{D}\\ \ell(Q)\le 2^{j_0-m}}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}} \\ &\le \lambda + m\varepsilon\lambda + \sum_{k\le j_0-m} g_k(x), \end{aligned}$$

where $g_k(x)$ is defined as

$$g_k(x) = \sum_{x\in Q\in\mathcal{D}_k} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}} \quad \text{for } k \in \mathbb{Z}$$

and $\mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^k\}$. Observe that only one nonzero term exists for each x in the previous sum. Let $\eta = \frac{\beta-(n-\alpha)}{2}$, we remark that since $\beta \in (n-\alpha, n]$ we have $\eta > 0$. We claim

$$S_{Q'} \subseteq \bigcup_{k\le j_0-m} \{x \in Q' : g_k(x) > \lambda(1-m\varepsilon)2^{\eta(k-j_0+m)}(1-2^{-\eta})\}. \tag{6.47}$$

To this end, observe that if $x \in S_{Q'}$, we have

$$2\lambda < I_\alpha^{\mathcal{D}}\mu(x) \le \lambda + m\varepsilon\lambda + \sum_{k\le j_0-m} g_k(x)$$

and thus,

$$\lambda(1-m\varepsilon) < \sum_{k\le j_0-m} g_k(x). \tag{6.48}$$

If we assume that

$$g_k(x) \le \lambda(1-m\varepsilon)2^{\eta(k-j_0+m)}(1-2^{-\eta}),$$

for all $k \le j_0 - m$, then

$$\sum_{k\le j_0-m} g_k(x) \le \lambda(1-m\varepsilon)(1-2^{-\eta}) \sum_{k\le j_0-m} 2^{\eta(k-j_0+m)} = \lambda(1-m\varepsilon),$$

which contradicts (6.48). Thus, there exists $k \le j_0 - m$ such that

$$g_k(x) > \lambda(1-m\varepsilon)2^{\eta(k-j_0+m)}(1-2^{-\eta}),$$

and we conclude the proof of the claim.

Observe that for any $\gamma > 0$ and any $k \le j_0$, we have

$$\begin{aligned}
\mathcal{H}^{\beta,Q_0}_\infty(\{x \in Q' : g_k(x) > \gamma\}) \le & \frac{1}{\gamma}\int_{Q'} g_k \, d\mathcal{H}^{\beta,Q_0}_\infty \\
= & \frac{1}{\gamma}\sum_{\substack{x\in Q\in\mathcal{D}_k\\ Q\subseteq Q'}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}}\ell(Q)^\beta \\
= & \frac{1}{\gamma}\sum_{\substack{x\in Q\in\mathcal{D}_k\\ Q\subseteq Q'}} \ell(Q)^{\beta-(n-\alpha)}\mu(Q) \\
\le & \frac{1}{\gamma}2^{2\eta k}\mu(Q') \\
= & \frac{1}{\gamma}2^{2\eta k}\frac{1}{\ell(Q')^{2\eta}}\ell(Q')^\beta\frac{\mu(Q')}{\ell(Q')^{n-\alpha}} \\
\le & \frac{1}{\gamma}2^{2\eta(k-j_0)}\ell(Q')^\beta \inf_{z\in Q'}\mathcal{M}_\alpha\mu(z) \\
\le & \frac{1}{\gamma}2^{2\eta(k-j_0)}\ell(Q')^\beta \mathcal{M}_\alpha\mu(x_{Q'}) \\
\le & \frac{1}{\gamma}2^{2\eta(k-j_0)}\ell(Q')^\beta\varepsilon\lambda, \qquad (6.49)
\end{aligned}$$

where $x_{Q'} \in Q'$ such that $\mathcal{M}_\alpha\mu(x_{Q'}) \le \varepsilon\lambda$, that we know exists because otherwise, $S_{Q'}$ would be empty. The combination of (6.47) and (6.49) then yields that

$$\begin{aligned}
\mathcal{H}^{\beta,Q_0}_\infty(S_{Q'}) \le & \sum_{k\le j_0-m}\mathcal{H}^{\beta,Q_0}_\infty(\{x\in Q' : g_k(x) > \lambda(1-m\varepsilon)2^{\eta(k-j_0+m)}(1-2^{-\eta})\}) \\
\le & \frac{1}{\lambda(1-m\varepsilon)(1-2^{-\eta})}\ell(Q')^\beta\varepsilon\lambda\sum_{k\le j_0-m}\frac{2^{2\eta(k-j_0)}}{2^{\eta(k-j_0+m)}} \\
= & \ell(Q')^\beta\frac{\varepsilon}{1-m\varepsilon}\frac{2^{-2\eta m}}{1-2^{-\eta}}\sum_{k\le j_0-m}2^{-\eta(j_0-m-k)} \\
= & \ell(Q')^\beta\frac{\varepsilon}{1-m\varepsilon}\frac{2^{-2\eta m}}{1-2^{-\eta}}\sum_{k=0}^{\infty}2^{-\eta k} \\
= & C_\eta 2^{-2\eta m}\frac{\varepsilon}{1-m\varepsilon}\ell(Q')^\beta.
\end{aligned}$$

We remark that the previous sum is finite because $\eta > 0$. Now we choose $m \in \mathbb{N}\cup\{0\}$ such that

$$\frac{1}{\varepsilon} - 2 \le m < \frac{1}{\varepsilon} - 1,$$

then $\frac{\varepsilon}{1-m\varepsilon} < 1$, and hence, there exists $c = c(n,\alpha,\beta)$ such that

$$\mathcal{H}^{\beta,Q_0}_\infty(S_{Q'}) \le C_\eta 2^{-\frac{2\eta}{\varepsilon}}\ell(Q')^\beta = C_\eta e^{-\frac{c}{\varepsilon}}\ell(Q')^\beta.$$

Using the previous estimate and the packing condition of the family $\{Q_{j_k}^{\lambda}\}_k$, we have

$$\begin{aligned}
\mathcal{H}_{\infty}^{\beta,Q_0}(S) \leq & \sum_{k:Q_{j_k}^{\lambda}\not\subseteq\tilde{Q}_m} \mathcal{H}_{\infty}^{\beta,Q_0}(S_{Q_{j_k}^{\lambda}}) + \sum_k \mathcal{H}_{\infty}^{\beta,Q_0}(S_{\tilde{Q}_k}) \\
\leq & C_\eta e^{-\frac{c}{\varepsilon}} \left(\sum_{k:Q_{j_k}^{\lambda}\not\subseteq\tilde{Q}_m} \ell(Q_{j_k}^{\lambda})^{\beta} + \sum_k \ell(\tilde{Q}_k)^{\beta} \right) \\
\leq & 2C_\eta e^{-\frac{c}{\varepsilon}} \mathcal{H}_{\infty}^{\beta,Q_0} \left(\bigcup_j Q_j^{\lambda} \right) \\
= & 2C_\eta e^{-\frac{c}{\varepsilon}} \mathcal{H}_{\infty}^{\beta,Q_0}(G_\lambda) .
\end{aligned}$$

□

As a consequence of the previous good-lambda estimate, we give the proof of Theorem 6.39.

Proof of Theorem 6.39. We observe that it is enough to prove (6.41) and (6.42) for the dyadic Riesz potential $I_\alpha^{\mathcal{D}}$ with a constant independent of the dyadic grid $\mathcal{D}$. To show this, we use (6.44), and we obtain the desired inequality,

$$\begin{aligned}
\|I_\alpha \mu\|_{L^p(\mathcal{H}_{\infty}^{\beta})} \leq & c_{n,\alpha} \left\| \sum_{i\in\{0,\frac{1}{3}\}^n} I_\alpha^{\mathcal{D}^i}\mu \right\|_{L^p(\mathcal{H}_{\infty}^{\beta})} \\
\leq & c_\beta c_{n,\alpha} \sum_{i\in\{0,\frac{1}{3}\}^n} \|I_\alpha^{\mathcal{D}^i}\mu\|_{L^p(\mathcal{H}_{\infty}^{\beta})} \\
\leq & C p\, c_\beta c_{n,\alpha} \sum_{i\in\{0,\frac{1}{3}\}^n} \|\mathcal{M}_\alpha \mu\|_{L^p(\mathcal{H}_{\infty}^{\beta})} \\
= & C_{\beta,\alpha,n} p \|\mathcal{M}_\alpha \mu\|_{L^p(\mathcal{H}_{\infty}^{\beta})}.
\end{aligned}$$

Similarly, we can obtain (6.42) from the dyadic estimate.

Let us first assume that μ has compact support. We may also assume that $\|\mathcal{M}_\alpha\mu\|_{L^p(\mathcal{H}_{\infty}^{\beta})}$ is finite because otherwise, there is nothing to prove. Let $\mathcal{D}$ be a dyadic grid, using Theorem 6.45, and the subadditivity of $\mathcal{H}_{\infty}^{\beta}$ we obtain,

$$\begin{aligned}
\mathcal{H}_{\infty}^{\beta}(\{x\in\mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda\}) \leq & \mathcal{H}_{\infty}^{\beta}(\{x\in\mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda, \mathcal{M}_\alpha\mu(x) \leq \varepsilon\lambda\}) \\
& + \mathcal{H}_{\infty}^{\beta}(\{x\in\mathbb{R}^n : \mathcal{M}_\alpha\mu(x) > \varepsilon\lambda\}) \\
\leq & C e^{-\frac{c}{\varepsilon}} \mathcal{H}_{\infty}^{\beta}(\{x\in\mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > \lambda\}) \\
& + C\mathcal{H}_{\infty}^{\beta}(\{x\in\mathbb{R}^n : \mathcal{M}_\alpha\mu(x) > \varepsilon\lambda\}),
\end{aligned}$$

for all $\lambda > 0$ and all $0 < \varepsilon < 1$. Let $N > 0$, we have

$$\begin{aligned}\int_0^N \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : I^{\mathcal{D}}_\alpha\mu(x) > \lambda\})d\lambda =& 2^p\int_0^{\frac{N}{2}} \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : I^{\mathcal{D}}_\alpha\mu(x) > 2\lambda\})d\lambda\\ \leq& 2^p Ce^{-\frac{c}{\varepsilon}}\int_0^{\frac{N}{2}} \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : I^{\mathcal{D}}_\alpha\mu(x) > \lambda\})d\lambda\\ &+ 2^p C\int_0^{\frac{N}{2}} \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : \mathcal{M}_\alpha\mu(x) > \varepsilon\lambda\})d\lambda\\ \leq& 2^p Ce^{-\frac{c}{\varepsilon}}\int_0^{N} \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : I^{\mathcal{D}}_\alpha\mu(x) > \lambda\})d\lambda\\ &+ \frac{2^p C}{\varepsilon^p}\int_0^{\frac{N\varepsilon}{2}} \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : \mathcal{M}_\alpha\mu(x) > \lambda\})d\lambda,\end{aligned}$$

for all $0 < \varepsilon < 1$. Notice that the first term is finite because μ has compact support. Let B be a ball such that $\operatorname{supp}\mu \subseteq B$. If $x \notin 2B$, then $|x-y| \geq r(B)$ for all $y \in B$. Hence,

$$I_\alpha\mu(x) = \gamma_{n,\alpha}\int_B \frac{d\mu(y)}{|x-y|^{n-\alpha}} \leq \gamma_{n,\alpha}\frac{\mu(B)}{r(B)^{n-\alpha}} \leq \gamma_{n,\alpha}\mathcal{M}_\alpha\mu(x),$$

for all $x \notin 2B$. Using the previous estimate, we obtain

$$\begin{aligned}\{x\in\mathbb{R}^n : I_\alpha\mu(x) > \lambda\} =& \{x\in 2B : I_\alpha\mu(x) > \lambda\}\cup\{x\in(2B)^c : I_\alpha\mu(x) > \lambda\}\\ \subseteq& \{x\in 2B : I_\alpha\mu(x) > \lambda\}\cup\{x\in(2B)^c : \mathcal{M}_\alpha\mu(x) > \gamma^{-1}_{n,\alpha}\lambda\},\end{aligned}$$

for all $\lambda > 0$. Then, we have

$$\begin{aligned}\int_0^N \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : I_\alpha\mu(x) > \lambda\})d\lambda \leq& \int_0^N \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in 2B : I_\alpha\mu(x) > \lambda\})d\lambda\\ &+ \int_0^N \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : \mathcal{M}_\alpha\mu(x) > \gamma^{-1}_{n,\alpha}\lambda\})d\lambda\\ \leq& r(2B)^\beta\frac{N^p}{p} + \gamma^p_{n,\alpha}\|\mathcal{M}_\alpha\mu\|^p_{L^p(\mathcal{H}^\beta_\infty)}\\ <&\infty,\end{aligned}$$

and using (6.43) we obtain that the dyadic counterpart is also finite. We take ε such that $2^pCe^{-\frac{c}{\varepsilon}} = \frac{1}{2}$, that is $\varepsilon = \frac{c}{\log(2^{p+1}C)}$. Then, since the right-hand side is finite, we can absorb it into the left-hand side, and we obtain

$$\int_0^N \lambda^{p-1}\mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : I^{\mathcal{D}}_\alpha\mu(x) > \lambda\})d\lambda \leq 2^{p+1}C\frac{\log(2^{p+1}C)^p}{c^p p}\int_{\mathbb{R}^n}|\mathcal{M}_\alpha\mu|^p\, d\mathcal{H}^\beta_\infty.$$

We obtain the desired inequality letting $N \to \infty$,

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|I^{\mathcal{D}}_\alpha\mu|^p\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}} \leq& c_n\frac{\log(2^{p+1}C)}{c}\left(\int_{\mathbb{R}^n}|\mathcal{M}_\alpha\mu|^p\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}\\ \leq& \tilde{C}\, p\left(\int_{\mathbb{R}^n}|\mathcal{M}_\alpha\mu|^p\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}},\end{aligned}$$

where $\tilde{C} = \tilde{C}(n, \beta, \alpha) > 0$.

Let us consider now the general case. Given a measure μ, we define $\mu_R = \mu\chi_{B(0,R)}$. Since μ_R has compact support, we have

$$\|I_\alpha^{\mathcal{D}}\mu_R\|_{L^p(\mathcal{H}^\beta_\infty)} \leq C\, p\|\mathcal{M}_\alpha\mu_R\|_{L^p(\mathcal{H}^\beta_\infty)} \leq C\, p\|\mathcal{M}_\alpha\mu\|_{L^p(\mathcal{H}^\beta_\infty)}.$$

Since $\mathcal{H}^\beta_\infty$ is continuous from below, we may apply the monotone convergence theorem (see **[PS23]** for instance), and we obtain the desired inequality.

In order to prove (6.42), let us first assume that μ has compact support. We may assume that $\|\mathcal{M}_\alpha\mu\|_{L^{p,\infty}(\mathcal{H}^\beta_\infty)} < \infty$, because otherwise there is nothing to prove. Let $N > 0$, using again Theorem 6.45, multiplying by λ^p and taking the supremum over $0 < \lambda \leq N$ we obtain the following estimate,

$$\begin{aligned}\sup_{0<\lambda\leq N} \lambda^p\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > 2\lambda\}) \leq & Ce^{-\frac{c}{\varepsilon}} \sup_{0<\lambda\leq N} \lambda^p\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > \lambda\}) \\ & + C \sup_{0<\lambda\leq N} \lambda^p\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : \mathcal{M}_\alpha\mu(x) > \varepsilon\lambda\}).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2^p}\sup_{0<\lambda\leq 2N} \lambda^p\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > \lambda\}) \leq & Ce^{-\frac{c}{\varepsilon}} \sup_{0<\lambda\leq 2N} \lambda^p\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > \lambda\}) \\ & + \frac{C}{\varepsilon^p}\|\mathcal{M}_\alpha\mu\|^p_{L^{p,\infty}(\mathcal{H}^\beta_\infty)}.\end{aligned}$$

Observe that the first term is finite because μ has compact support. Let B be a ball such that $\mathrm{supp}\mu \subseteq B$. Using the same argument as before, we have

$$\begin{aligned}\sup_{0<\lambda\leq 2N} \lambda^p\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha\mu(x) > \lambda\}) \leq & \sup_{0<\lambda\leq 2N} \lambda^p\mathcal{H}^\beta_\infty(\{x \in 2B : I_\alpha\mu(x) > \lambda\}) \\ & + \sup_{0<\lambda\leq 2N} \lambda^p\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : \mathcal{M}_\alpha\mu(x) > \gamma_{n,\alpha}^{-1}\lambda\}) \\ \leq & (2N)^p r(2B)^\beta + \gamma^p_{n,\alpha}\|\mathcal{M}_\alpha\mu\|^p_{L^{p,\infty}(\mathcal{H}^\beta_\infty)} \\ < & \infty,\end{aligned}$$

and thus, using (6.43), we obtain that the dyadic counterpart is also finite.

We take again ε such that $2^p Ce^{-\frac{c}{\varepsilon}} = \frac{1}{2}$, that is $\varepsilon = \frac{c}{\log(2^{p+1}C)}$. Then, since the right-hand side is finite, we can absorb it into the left-hand side, and we obtain

$$\begin{aligned}\sup_{0<\lambda\leq 2N} \lambda\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha^{\mathcal{D}}\mu(x) > \lambda\})^{\frac{1}{p}} \leq & c_n\frac{\log(2^{p+1}C)}{c}\|\mathcal{M}_\alpha\mu\|_{L^{p,\infty}(\mathcal{H}^\beta_\infty)} \\ \leq & C' p\|\mathcal{M}_\alpha\mu\|_{L^{p,\infty}(\mathcal{H}^\beta_\infty)}.\end{aligned}$$

Letting $N \to \infty$, we obtain the desired inequality. The case of a general measure μ is obtained in the same way as before. □

Using Proposition 6.43, it is easy to see that (6.45) implies the following good lambda inequality for the Riesz potential.

Corollary 6.45. *Let $0 < \alpha < n \in \mathbb{N}$, $\beta \in (n-\alpha, n]$ and μ be a locally finite measure in $\mathbb{R}^n$. Then, there exist constants $C, c > 0$ depending on n, α, β such that*

$$\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha \mu(x) > 2^{n+1}\gamma_1\lambda, \mathcal{M}_\alpha \mu(x) \le \varepsilon\lambda\}) \le Ce^{-\frac{c}{\varepsilon}}\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha\mu(x) > \gamma_2^{-1}\lambda\}), \tag{6.50}$$

for all $\lambda > 0$ and all $0 < \varepsilon < 1$, where $\gamma_1, \gamma_2 > 1$ are the constants in (6.44) *and* (6.43)*, respectively.*

6.5.1 Exponential integrability of Riesz potentials for Morrey measures

As a consequence of the good-lambda inequality with exponential decay (6.50), we can prove the following result about the exponential integrability of $I_\alpha\mu$ for measures in the local Morrey space $\mathscr{M}^{n-\alpha}(B)$. This is the natural endpoint $p = \infty$ in Theorem 6.39. We give an alternative proof of a result of Adams and Xiao **[AX15]** (see also **[Che25, Theorem 1.7]**).

Theorem 6.46. *Let $0 < \alpha < n \in \mathbb{N}$, $\beta \in (n-\alpha, n]$ and μ be a locally finite measure in $\mathbb{R}^n$. Then there exist $C = C(n, \alpha, \beta) > 0$ and $\gamma = \gamma(n, \alpha, \beta) > 0$ such that*

$$\frac{1}{r(2B)^\beta}\int_{2B} \exp\left(\frac{\gamma\, I_\alpha(\mu\chi_B)(x)}{\|\mu\|_{\mathscr{M}^{n-\alpha}(B)}}\right) d\mathcal{H}^\beta_\infty \le C,$$

for every ball B such that $0 < \|\mu\|_{\mathscr{M}^{n-\alpha}(B)} < \infty$.

Here, $\mu\chi_B$ denotes the restriction of the measure μ to B, and

$$\|\mu\|_{\mathscr{M}^{n-\alpha}(B)} := \sup_Q \frac{\mu(Q \cap B)}{\ell(Q)^{n-\alpha}},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

Proof. Let B be a ball in $\mathbb{R}^n$ such that $\|\mu\|_{\mathscr{M}^{n-\alpha}(B)} < \infty$. By homogeneity, replacing μ by $\mu/\|\mu\|_{\mathscr{M}^{n-\alpha}(B)}$, it is enough to prove the result under the normalization $\|\mu\|_{\mathscr{M}^{n-\alpha}(B)} \le 1$. Observe that $r(B) \le |x-y|$ for any $y \in B$ and $x \notin 2B$, hence

$$I_\alpha(\mu\chi_B)(x) = \gamma_{n,\alpha}\int_B \frac{d\mu(y)}{|x-y|^{n-\alpha}} \le \gamma_{n,\alpha}\frac{\mu(B)}{r(B)^{n-\alpha}} \le C_n\gamma_{n,\alpha}\,\|\mu\|_{\mathscr{M}^{n-\alpha}(B)} \le C_n\gamma_{n,\alpha},$$

for each $x \notin 2B$.

Therefore, if $\lambda > C_n\gamma_{n,\alpha}$ we have $\{x \in \mathbb{R}^n : I_\alpha(\mu\chi_B)(x) > \lambda\} \subseteq 2B$. For fixed $\lambda > \lambda_0 := \max\{1, C_n\gamma_2\gamma_{n,\alpha}\}$, applying (6.50) with $\varepsilon = \frac{1}{\lambda}$ we obtain

$$\begin{aligned}
\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha(\mu\chi_B)(x) > 2^{n+1}\gamma_1\lambda\}) &\le Ce^{-c\lambda}\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha(\mu\chi_B)(x) > \gamma_2^{-1}\lambda\}) \\
&\quad + \mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : \mathcal{M}_\alpha(\mu\chi_B)(x) > 1\}) \\
&= Ce^{-c\lambda}\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : I_\alpha(\mu\chi_B)(x) > \gamma_2^{-1}\lambda\}) \\
&\le Ce^{-c\lambda}r(2B)^\beta.
\end{aligned}$$

Using the previous estimate, we can conclude the proof,

$$\begin{aligned}
\int_{2B} \exp\left(\frac{c\, I_\alpha(\mu\chi_B)}{2^{n+2}\gamma_1}\right) d\mathcal{H}^\beta_\infty &= \mathcal{H}^\beta_\infty(2B) \\
&\quad + \frac{c}{2^{n+2}\gamma_1}\int_0^\infty e^{\frac{c}{2^{n+2}\gamma_1}\lambda}\mathcal{H}^\beta_\infty(\{x\in 2B : I_\alpha(\mu\chi_B)(x) > \lambda\})d\lambda \\
&\le Cr(2B)^\beta + \frac{c}{2}\int_0^\infty e^{\frac{c}{2}\lambda}\mathcal{H}^\beta_\infty(\{x\in 2B : I_\alpha(\mu\chi_B)(x) > 2^{n+1}\gamma_1\lambda\})d\lambda \\
&= Cr(2B)^\beta + \frac{c}{2}\int_0^{\lambda_0} e^{\frac{c}{2}\lambda}\mathcal{H}^\beta_\infty(\{x\in 2B : I_\alpha(\mu\chi_B)(x) > 2^{n+1}\gamma_1\lambda\})d\lambda \\
&\quad + \frac{c}{2}\int_{\lambda_0}^\infty e^{\frac{c}{2}\lambda}\mathcal{H}^\beta_\infty(\{x\in 2B : I_\alpha(\mu\chi_B)(x) > 2^{n+1}\gamma_1\lambda\})d\lambda \\
&\le Cr(2B)^\beta + \frac{c}{2}e^{\frac{c}{2}\lambda_0}\lambda_0\, r(2B)^\beta + C\frac{c}{2}r(2B)^\beta\int_{\lambda_0}^\infty e^{-\frac{c}{2}\lambda}d\lambda \\
&= C'r(2B)^\beta.
\end{aligned}$$

□

An application to PDEs: exponential integrability of gradient estimates for p-Laplace type equation with measure data

In this section, we use Theorem 6.46 to prove the capacitary local exponential integrability of the gradient of weak solutions of singular p-Laplace type equations with measure data. We consider the quasilinear elliptic equation

$$-\operatorname{div}(A(x,\nabla u)) = \mu, \tag{6.51}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$, where $n\ge 2$. Here, μ is a finite Radon measure in Ω. The continuous vector field $A = (A_1,...,A_n) : \Omega\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is C^1 in the gradient variable z, and it satisfies the following conditions:

$$\begin{cases}
|A(x,z)| + |D_z A(x,z)|(|z|^2+s^2)^{\frac{1}{2}} \le L(|z|^2+s^2)^{\frac{p-1}{2}}, \\
\nu^{-1}(|z|^2+s^2)^{\frac{p-2}{2}}|\lambda|^2 \le \langle D_z A(x,z)\lambda,\lambda\rangle, \\
|A(x,z)-A(x_0,z)| \le L_1\omega(|x-x_0|)(|z|^2+s^2)^{\frac{p-1}{2}},
\end{cases} \tag{6.52}$$

for all $x, x_0\in\Omega$ and $z,\lambda\in\mathbb{R}^n$. The parameters ν, L, s, L_1 are fixed and such that $0<\nu\le L$ and $s\ge 0$, $L_1\ge 0$, the function $\omega : [0,\infty)\longrightarrow[0,1]$ is a non-decreasing subadditive function satisfying

$$\lim_{r\downarrow 0}\omega(r) = \omega(0) = 0, \tag{6.53}$$

and the Dini condition

$$\int_0^1 \omega(r)\frac{dr}{r} < \infty. \tag{6.54}$$

The model example of (6.51) is the p-Laplacian equation

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu \qquad \text{in } \Omega.$$

We refer to **[DM10]** for a more detailed explanation of the setting.

The gradient estimates of the weak solutions of (6.51) for $p \geq 2$ have been well studied in the literature **[DM10; DM11]** in terms of nonlinear potentials. We are interested in the singular case $p \in (1,2)$. In the seminal work **[DM10]** the authors prove the following gradient estimate for $p \in \left(2-\frac{1}{n}, 2\right)$, if $u \in C^1(\Omega)$ solves (6.51), then there exists $C = C(n,p,\lambda,\omega) > 0$ such that

$$|\nabla u(x)| \leq C I_1(\mu \chi_{B(x,R)})(x)^{\frac{1}{p-1}} + C \frac{1}{|B(x,R)|} \int_{B(x,R)} (|\nabla u(y)| + s) dy, \tag{6.55}$$

for each ball $B(x,R) \subset \Omega$ with $R \in (0,1]$. Later, the case $p \in \left(\frac{3n-2}{2n-1}, 2-\frac{1}{n}\right)$ was studied in **[NP20]**, and recently improved in **[DZ24]**.

We can prove the following exponential integrability result using Theorem 6.46 and the gradient estimate (6.55).

Theorem 6.47. *Let $\beta \in (n-1, n]$, $p \in \left(2-\frac{1}{n}, 2\right)$ and let $u \in C^1(\Omega)$ be a weak solution of* (6.51) *under the assumptions* (6.52)–(6.54)*. Then, there exist $C_1, C_2, C_3 > 0$ such that*

$$\frac{1}{r(B)^\beta} \int_B \exp\left(C_1 \frac{|\nabla u|^{p-1}}{\|\mu\|_{\mathscr{M}^{n-1}(2B)}} \right) d\mathcal{H}^\beta_\infty \leq C_2 \exp\left(\frac{C_3}{|2B|} \int_{2B} \frac{(|\nabla u(y)| + s)}{\|\mu\|_{\mathscr{M}^{n-1}(2B)}} dy \right)^{p-1}, \tag{6.56}$$

for all balls B such that $2B \subset \Omega$ and $0 < \|\mu\|_{\mathscr{M}^{n-1}(2B)} < \infty$.

Proof. The proof follows the approach of **[HJ12, Theorem 1.5]**. Let $B = B(x_0, R)$ be a ball such that $2B \subset \Omega$. It follows from (6.55) that

$$|\nabla u(x)|^{p-1} \leq C I_1(\mu \chi_{2B})(x) + C \left(\frac{1}{|2B|} \int_{2B} (|\nabla u(y)| + s) dy \right)^{p-1}, \tag{6.57}$$

for all $x \in B$. Applying Theorem 6.46 there exist $C' = C'(n,\beta) > 0$ and $\gamma = \gamma(n,\beta) > 0$ such that

$$\frac{1}{r(2B)^\beta} \int_{2B} \exp\left(\gamma \frac{I_1 \mu(x)}{\|\mu\|_{\mathscr{M}^{n-1}(2B)}} \right) d\mathcal{H}^\beta_\infty \leq C'.$$

Hence,

$$\begin{aligned} \frac{1}{r(B)^\beta} \int_B \exp\left(\frac{\gamma}{C} \frac{|\nabla u(x)|^{p-1}}{\|\mu\|_{\mathscr{M}^{n-1}(2B)}} \right) d\mathcal{H}^\beta_\infty \leq & C \left(\frac{1}{r(2B)^\beta} \int_{2B} \exp\left(\gamma \frac{I_1 \mu(x)}{\|\mu\|_{\mathscr{M}^{n-1}(2B)}} \right) d\mathcal{H}^\beta_\infty \right) \\ & \cdot \exp\left(\frac{\gamma}{|2B|} \int_{2B} \frac{(|\nabla u(y)| + s)}{\|\mu\|_{\mathscr{M}^{n-1}(2B)}} dy \right)^{p-1} \\ \leq & C_2 \exp\left(\frac{\gamma}{|2B|} \int_{2B} \frac{(|\nabla u(y)| + s)}{\|\mu\|_{\mathscr{M}^{n-1}(2B)}} dy \right)^{p-1}. \end{aligned}$$

This concludes the proof of (6.56). □

Remark 6.48. We observe that our techniques cannot reach the endpoint $\beta = n-1$.

6.6 Capacitary Orlicz spaces and maximal functions

A function $B : [0,\infty) \to [0,\infty)$ is called a Young function if it is continuous, convex, and strictly increasing, satisfying $B(0) = 0$ and $\lim_{t\to\infty} B(t) = \infty$. Given two Young functions A and B, we write $A(t) \approx B(t)$ if there exist constants $t_0, c_1, c_2 > 0$ such that $c_1 A(t) \le B(t) \le c_2 A(t)$ for all $t \ge t_0$.

A Young function B is said to be doubling if $B(2t) \le CB(t)$ for all $t > 0$, and submultiplicative if there exists $C > 0$ such that $B(st) \le CB(s)B(t)$ for all $s, t > 0$. Standard examples include $B(t) = t^r$ for $r \ge 1$, and more generally $B(t) = t^a[\log(e+t)]^b$ with $a \ge 1$ and $b > 0$, both of which are submultiplicative.

Given a nonempty set $E \subset \mathbb{R}^n$ and a Young function B, we define the Orlicz space $L^B(E, \mathcal{H}^\beta_\infty)$ as the set of all functions f for which $B(|f|/\lambda)$ is integrable on E with respect to the Hausdorff content $\mathcal{H}^\beta_\infty$ for some $\lambda > 0$. This space is equipped with the Luxemburg quasinorm

$$\|f\|_{L^B(E,\mathcal{H}^\beta_\infty)} = \inf\left\{\lambda > 0 : \int_E B\left(\frac{|f|}{\lambda}\right) d\mathcal{H}^\beta_\infty \le 1\right\}.$$

In the specific case where E is a cube Q, we also consider the mean Luxemburg quasinorm, defined by

$$\|f\|_{B,Q,\mathcal{H}^\beta_\infty} = \inf\left\{\lambda > 0 : \frac{1}{\ell(Q)^\beta}\int_Q B\left(\frac{|f|}{\lambda}\right) d\mathcal{H}^\beta_\infty \le 1\right\}.$$

Remark 6.49. With the previous notation, Theorem 6.18 may be restated as

$$\|b - b_Q\|_{B,Q,\mathcal{H}^\beta_\infty} \le C\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)},$$

for each cube Q, where $B(t) = e^t - 1$.

Given a Young function B, its complementary Young function is defined as

$$\bar{B}(t) = \sup\{st - B(s) : s > 0\},$$

for every $t > 0$.

Remark 6.50. Let $B(t) = t\log(e+t)$, the complementary Young function $\bar{B}$ satisfies $\bar{B}(t) \lesssim e^t - 1$ (see **[BS88]**).

We need the following Hölder's inequality for Orlicz spaces with respect to the Hausdorff content.

Lemma 6.51. *Given a Young function B, then for all functions f and g and all cubes Q,*

$$\frac{1}{\ell(Q)^\beta}\int_Q |fg| d\mathcal{H}^\beta_\infty \le C_\beta \|f\|_{B,Q,\mathcal{H}^\beta_\infty}\|g\|_{\bar{B},Q,\mathcal{H}^\beta_\infty}. \tag{6.58}$$

Proof. We have Young's inequality,

$$xy \le B(x) + \bar{B}(y),$$

for all $x, y \ge 0$. Then, we have

$$\frac{|f(x)|}{\|f\|_{B,Q,\mathcal{H}^\beta_\infty}}\frac{|g(x)|}{\|g\|_{\bar{B},Q,\mathcal{H}^\beta_\infty}} \le B\left(\frac{|f(x)|}{\|f\|_{B,Q,\mathcal{H}^\beta_\infty}}\right) + \bar{B}\left(\frac{|g(x)|}{\|g\|_{\bar{B},Q,\mathcal{H}^\beta_\infty}}\right).$$

Integrating over Q,

$$\begin{aligned}\frac{1}{\ell(Q)^\beta}\int_Q \frac{|f|}{\|f\|_{B,Q,\mathcal{H}^\beta_\infty}}\frac{|g|}{\|g\|_{\bar{B},Q,\mathcal{H}^\beta_\infty}}d\mathcal{H}^\beta_\infty \leq & C_\beta \frac{1}{\ell(Q)^\beta}\int_Q B\left(\frac{|f(x)|}{\|f\|_{B,Q,\mathcal{H}^\beta_\infty}}\right)d\mathcal{H}^\beta_\infty \\ & + C_\beta \frac{1}{\ell(Q)^\beta}\int_Q \bar{B}\left(\frac{|g(x)|}{\|g\|_{\bar{B},Q,\mathcal{H}^\beta_\infty}}\right)d\mathcal{H}^\beta_\infty \\ \leq & 2C_\beta.\end{aligned}$$

□

These concepts motivate the following definition of the Orlicz fractional maximal operator associated with Hausdorff content.

Definition 6.52. Let $n \in \mathbb{N}$ and $0 < \alpha < \beta \leq n$. The β-dimensional fractional maximal operator with respect to a Young function B of a function f in $\mathbb{R}^n$ is defined by

$$\mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x) = \sup_Q \chi_Q(x)\ell(Q)^\alpha \|f\|_{B,Q,\mathcal{H}^\beta_\infty}.$$

When $\alpha = 0$, we simply write $\mathcal{M}_{0,B,\mathcal{H}^\beta_\infty} f = \mathcal{M}_{B,\mathcal{H}^\beta_\infty} f$. Let $\mathcal{D}$ be the collection of the usual dyadic cubes in $\mathbb{R}^n$, we define the dyadic β-dimensional fractional maximal operator with respect to a Young function B of a function f as

$$\mathcal{M}^{\mathcal{D}}_{\alpha,B,\mathcal{H}^{\beta,Q_0}_\infty} f(x) = \sup_{P\in\mathcal{D}} \chi_P(x)\ell(P)^\alpha \|f\|_{B,P,\mathcal{H}^{\beta,Q_0}_\infty}.$$

Remark 6.53. By setting $g = 1$ in the Hölder inequality, we observe that for any Young function B and $0 < \alpha < \beta$,

$$\mathcal{M}_{\alpha,\mathcal{H}^\beta_\infty} f(x) \leq C_\beta\, \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x),$$

for all $x \in \mathbb{R}^n$.

Lemma 6.54. *Let $0 < \beta \leq n$ and $0 \leq \alpha < \beta$. Let f be a function for which the quantities below are well defined, and let B be a Young function. Suppose that for some cube Q and some parameter $t > 0$, we have*

$$\ell(Q)^\alpha \|f\|_{B,Q,\mathcal{H}^\beta_\infty} > t. \tag{6.59}$$

Then, there exists a dyadic cube $P \in \mathcal{D}$ such that $Q \subset 3P$ and a constant $C_{\beta,n} > 0$ such that

$$\ell(P)^\alpha \|f\|_{B,P,\mathcal{H}^{\beta,Q_0}_\infty} > c_{\beta,n} t.$$

Proof. Let $k \in \mathbb{Z}$ be the unique integer such that $2^{k-1} < \ell(Q) \leq 2^k$. There exist at least one and at most 2^n dyadic cubes $\{P_j\}_{j=1}^m$ ($1 \leq m \leq 2^n$) of side length 2^k that intersect the interior of Q. Note that $Q \subset 3P_j$ for each j. We claim that one of these cubes, say P_1, satisfies

$$\ell(P_1)^\alpha \left\|\chi_{P_1} f\right\|_{B,Q,\mathcal{H}^{\beta,Q_0}_\infty} > \frac{t}{c_\beta 2^n}.$$

Indeed, if for each $j = 1, \dots, m$ we had

$$\ell(P_j)^\alpha \left\|\chi_{P_j} f\right\|_{B,Q,\mathcal{H}^\beta_\infty} \leq \frac{t}{c_\beta 2^n},$$

then it would follow that

$$\begin{aligned}
\ell(Q)^\alpha \|f\|_{B,Q,\mathcal{H}^{\beta,Q_0}_\infty} =& \ell(Q)^\alpha \left\|\chi_{\cup_{j=1}^m P_j} f\right\|_{B,Q,\mathcal{H}^\beta_\infty} \\
\leq& c_\beta \sum_{j=1}^m \ell(Q)^\alpha \left\|\chi_{P_j} f\right\|_{B,Q,\mathcal{H}^{\beta,Q_0}_\infty} \\
\leq& c_\beta \sum_{j=1}^m \ell(P_j)^\alpha \left\|\chi_{P_j} f\right\|_{B,Q,\mathcal{H}^{\beta,Q_0}_\infty} \\
\leq& c_\beta \sum_{j=1}^m \frac{t}{c_\beta 2^n} \\
=& \frac{m}{2^n} t \\
\leq& t,
\end{aligned}$$

which contradicts (6.59). Therefore, we must have

$$\ell(P_1)^\alpha \left\|\chi_{P_1} f\right\|_{B,Q,\mathcal{H}^{\beta,Q_0}_\infty} > \frac{t}{c_\beta 2^n}.$$

Using the fact that $\ell(P_1)^\beta \leq 2^\beta \ell(Q)^\beta$, we obtain

$$\begin{aligned}
\frac{t}{c_\beta 2^n} <& \ell(P_1)^\alpha \left\|\chi_{P_1} f\right\|_{B,Q,\mathcal{H}^{\beta,Q_0}_\infty} \\
\leq& 2^\beta \ell(P_1)^\alpha \|f\|_{B,P_1,\mathcal{H}^{\beta,Q_0}_\infty}.
\end{aligned}$$

This concludes the proof with the constant $C = \frac{1}{c_\beta 2^{n+\beta}}$. □

The following corollary is obtained by using the previous lemma together with an argument from **[COS24, Lemma 2.4]**.

Corollary 6.55. *There exists a constant $c = c(n, \beta) > 0$ such that*

$$\mathcal{H}^\beta_\infty\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x) > t\right\}\right) \leq 3^n \mathcal{H}^\beta_\infty\left(\left\{x \in \mathbb{R}^n : \mathcal{M}^{\mathcal{D}}_{\alpha,B,\mathcal{H}^{\beta,Q_0}_\infty} f(x) > c\, t\right\}\right)$$

holds for every $t > 0$.

Proof. For $t > 0$, we define

$$E_t = \left\{x \in \mathbb{R}^n : \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x) > t\right\},$$

and

$$L_t = \left\{x \in \mathbb{R}^n : \mathcal{M}^{\mathcal{D}}_{\alpha,B,\mathcal{H}^{\beta,Q_0}_\infty} f(x) > t\right\}.$$

Let $x \in E_t$, then there exists a cube Q_x with $x \in Q_x$ such that $\ell(Q)^\alpha \|f\|_{B,Q,\mathcal{H}^\beta_\infty} > t$, and this implies $Q_x \subseteq E_t$. By Lemma 6.54, there exists $P_x \in \mathcal{D}$ with $Q_x \subseteq 3P_x$ and such that $\ell(P_x)^\alpha \|f\|_{B,P_x,\mathcal{H}^{\beta,\mathcal{Q}_0}_\infty} > c\,t$, where $c = c(n,\beta) > 0$. Hence, $P_x \subseteq L_{c\,t}$ and

$$E_t \subseteq \bigcup_{x\in E_t} Q_x \subseteq \bigcup_{x\in E_t} 3P_x \subseteq \{3P : P \in \mathcal{D} \text{ and } P \subseteq L_{c\,t}\}\,.$$

For each $P \in \mathcal{D}$ with $P \subseteq L_{c\,t}$ we can write $3P = \bigcup_{i=1}^{3^n} \tau_i P$ for appropriate translation operators $\{\tau_i\}_{i=1}^{3^n}$. Hence, using the subadditivity and translation invariance of $\mathcal{H}^\beta_\infty$, we have

$$\begin{aligned}
\mathcal{H}^\beta_\infty(E_t) \leq& \mathcal{H}^\beta_\infty(\{3P : P \in \mathcal{D} \text{ and } P \subseteq L_{c\,t}\})\\
=& \mathcal{H}^\beta_\infty\left(\left\{\cup_{i=1}^{3^n}\tau_i P : P \in \mathcal{D} \text{ and } P \subseteq L_{c\,t}\right\}\right)\\
\leq& \sum_{i=1}^{3^n} \mathcal{H}^\beta_\infty(\{\tau_i P : P \in \mathcal{D} \text{ and } P \subseteq L_{c\,t}\})\\
=& 3^n \mathcal{H}^\beta_\infty(\{P : P \in \mathcal{D} \text{ and } P \subseteq L_{c\,t}\})\\
=& 3^n \mathcal{H}^\beta_\infty(L_{c\,t})\,.
\end{aligned}$$

□

Definition 6.56. Given a Young function B, we define the function h_B by

$$h_B(s) = \sup_{t>0} \frac{B(st)}{B(t)},$$

for each $0 \leq s < \infty$.

Remark 6.57. As shown in **[CF03, Lemma 3.11]**, the function h_B is submultiplicative, increasing on $[0,\infty)$, and strictly increasing on $[0,1]$, with $h_B(1) = 1$.

We also use the following lemma, whose proof can be found in **[HLP88]**.

Lemma 6.58. *Suppose that the function $t \mapsto \Phi(t)/t$ is decreasing. Then, for any sequence of positive numbers $\{x_k\}$, we have*

$$\Phi\left(\sum_k x_k\right) \leq \sum_k \Phi(x_k).$$

With the preceding results at hand, we can now establish the following theorem, which will be essential for deriving the modular weak-type endpoint estimate stated in Theorem 6.69.

Theorem 6.59. *Let $0 < \beta \leq n$ and $0 < \alpha < \beta$. Let B be a Young function such that $t \mapsto B(t)/t^{\beta/\alpha}$ is decreasing on $(0,\infty)$. Then, there exists a constant $C = C(n,\beta,\alpha,B) > 0$ such that for all $t > 0$, the operator $\mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty}$ satisfies the modular weak-type inequality*

$$\Phi\left(\mathcal{H}^\beta_\infty\big(\{x \in \mathbb{R}^n : \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x) > t\}\big)\right) \leq C \int_{\mathbb{R}^n} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^\beta_\infty,$$

for all non-negative functions $f \in L^B(\mathbb{R}^n, \mathcal{H}^\beta_\infty)$. Here, Φ denotes any function satisfying

$$\Phi(s) \leq C_1\Phi_1(s) = \begin{cases} 0 & \text{if } s = 0, \\ \dfrac{s}{h_B(s^{\alpha/\beta})} & \text{if } s > 0. \end{cases}$$

Remark 6.60. The function Φ_1 is well defined; indeed, by **[CF03, Lemma 3.12]**, since $B(t)/t^{\beta/\alpha}$ is decreasing, it follows that $0 < h_B(s^{\alpha/\beta}) < \infty$ for all $s > 0$. Furthermore, the function Φ_1 is increasing and $\Phi_1(t)/t$ is decreasing. Moreover, there exists an invertible function Φ such that $\Phi(s) \leq C_1\Phi_1(s)$.

Proof. Fix $t > 0$, and define the sets

$$E_t = \{x \in \mathbb{R}^n : \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x) > t\},$$

and

$$L_t = \left\{x \in \mathbb{R}^n : \mathcal{M}^{\mathcal{D}}_{\alpha,B,\mathcal{H}^{\beta,Q_0}_\infty} f(x) > t\right\}.$$

By Corollary 6.55 we have

$$\mathcal{H}^\beta_\infty(E_t) \leq 3^n \mathcal{H}^\beta_\infty(L_{At}),$$

where $A = \frac{1}{c_{n,\beta}3^n}$. Using that Φ_1 is increasing and (6.4), we have

$$\Phi_1(\mathcal{H}^\beta_\infty(E_t)) \leq \Phi_1(C_{\beta,n}\mathcal{H}^{\beta,Q_0}_\infty(L_{At})).$$

We may write

$$L_{At} = \bigcup_j P_j,$$

where $\{P_j\}_j$ is the maximal collection of disjoint dyadic cubes contained in L_{At}. Observe that each P_j satisfies

$$\ell(P_j)^\alpha \|f\|_{B,P_j,\mathcal{H}^{\beta,Q_0}_\infty} > At.$$

By the definition of the Luxemburg norm, this implies

$$\|(At)^{-1}\ell(P_j)^\alpha f\|_{B,P_j,\mathcal{H}^{\beta,Q_0}_\infty} > 1,$$

for each j. Consequently,

$$\begin{aligned} 1 &< \frac{1}{\ell(P_j)^\beta} \int_{P_j} B\left(\frac{\ell(P_j)^\alpha f(x)}{At}\right) d\mathcal{H}^{\beta,Q_0}_\infty \\ &\leq \frac{h_B(A^{-1}\ell(P_j)^\alpha)}{\ell(P_j)^\beta} \int_{P_j} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^{\beta,Q_0}_\infty \\ &\leq h_B(A^{-1}C_\beta^{-\alpha/\beta}) \frac{h_B((C_\beta\ell(P_j)^\beta)^{\alpha/\beta})}{\ell(P_j)^\beta} \int_{P_j} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^{\beta,Q_0}_\infty \\ &= h_B(A^{-1}C_\beta^{-\alpha/\beta}) \frac{1}{\Phi_1(C_\beta\ell(P_j)^\beta)} \int_{P_j} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^{\beta,Q_0}_\infty, \end{aligned}$$

where C_β is the constant from (6.4). Thus,

$$\Phi_1(C_\beta\ell(P_j)^\beta) < h_B(A^{-1}C_\beta^{-\alpha/\beta}) \int_{P_j} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^{\beta,Q_0}_\infty, \tag{6.60}$$

for each j. Invoking Lemma 6.7, we obtain a subfamily $\{P_{j_v}\}_v$ of $\{P_j\}$ and a family of non-overlapping ancestors $\widetilde{P}_v$ satisfying:

1.
$$L_{At} = \bigcup_j P_j \subset \left(\bigcup_v P_{j_v}\right) \cup \left(\bigcup_v \widetilde{P}_v\right).$$

2.
$$\sum_{P_{j_v} \subset P} \ell(P_{j_v})^\beta \leq 2\ell(P)^\beta, \quad \text{for each dyadic cube } P \text{ (with respect to } \mathcal{D}_k).$$

3.
$$\begin{aligned}\mathcal{H}^{\beta,Q_0}_\infty\left(\bigcup_j P_j\right) &\leq \sum_v \ell(P_{j_v})^\beta + \sum_v \ell(\widetilde{P}_v)^\beta \\ &\leq \sum_v \ell(P_{j_v})^\beta.\end{aligned}$$

Therefore, combining Lemma 6.58 and (6.60), we deduce that

$$\begin{aligned}\Phi_1(C_\beta \mathcal{H}^{\beta,Q_0}_\infty(L_{At})) &= \Phi_1(C_\beta \mathcal{H}^{\beta,Q_0}_\infty(\cup_j P_j)) \\ &\leq \Phi_1\left(\sum_v C_\beta \ell(P_{j_v})^\beta\right) \\ &\leq \sum_v \Phi_1(C_\beta \ell(P_{j_v})^\beta) \\ &\leq h_B(A^{-1}C_\beta^{-\alpha/\beta}) \sum_v \int_{P_{j_v}} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^{\beta,Q_0}_\infty \\ &\leq 2\, h_B(A^{-1}C_\beta^{-\alpha/\beta}) \int_{\mathbb{R}^n} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^{\beta,Q_0}_\infty,\end{aligned}$$

where the last inequality follows from the packing condition satisfied by $\{P_{j_v}\}_v$ (see (6.8)). We conclude:

$$\begin{aligned}\Phi_1(\mathcal{H}^\beta_\infty(E_t)) &\leq \Phi_1(C_{\beta,n}\mathcal{H}^{\beta,Q_0}_\infty(L_{At})) \\ &\leq 2\, h_B(A^{-1}C_\beta^{-\alpha/\beta}) \int_{\mathbb{R}^n} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^{\beta,Q_0}_\infty \\ &\leq C_\beta h_B(A^{-1}C_\beta^{-\alpha/\beta}) \int_{\mathbb{R}^n} B\left(\frac{f(x)}{t}\right) d\mathcal{H}^\beta_\infty.\end{aligned}$$

□

6.7 Characterization of BMO^β in terms of commutator of fractional integrals

The classical theorem of Coifman, Rochberg, and Weiss **[CRW76]** asserts that for any Calderón–Zygmund operator T and any locally integrable function b, the commutator

$$[b,T]f := bTf - T(bf)$$

is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. This result not only characterizes the BMO space but also reveals its intrinsic connection with the structure of singular integrals. A natural extension of this problem concerns commutators of fractional integral operators. Chanillo **[Cha82]** proved that the corresponding Riesz commutator

$$[b, I_\alpha]f = b\, I_\alpha f - I_\alpha(bf)$$

is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, with $1 < p < \frac{n}{\alpha}$ and

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n},$$

if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Moreover,

$$\|[b, I_\alpha] : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)\| \simeq \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Motivated by this result, the aim of this section is to extend the classical Riesz commutator characterization of Chanillo **[Cha82]** to the capacitary setting. Specifically, we prove that membership of the symbol b in $\mathrm{BMO}^\beta(\mathbb{R}^n)$ is equivalent to the boundedness of the Riesz commutator $[b, I_\alpha]$ between suitable capacitary Lebesgue spaces associated with the Hausdorff content $\mathcal{H}^\beta_\infty$. This recovers the Euclidean theory when $\beta = n$.

Theorem 6.61. *Let $0 < \beta \le n$ and $0 < \alpha < \beta$. For $1 < p < \frac{\beta}{\alpha}$, define the exponent q by the relation*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\beta}.$$

Let b be a measurable function such that the commutator $[b, I_\alpha]$ is well defined. Then, the following statements are equivalent:

(i) $b \in \mathrm{BMO}^\beta(\mathbb{R}^n)$.

(ii) There exists a constant $C = C(\alpha, \beta, n, p) > 0$ such that

$$\left(\int_{\mathbb{R}^n} |[b, I_\alpha]f|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f|^p \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}, \tag{6.61}$$

for all bounded functions f with compact support.

Furthermore, the norms satisfy the equivalence

$$\|[b, I_\alpha]\|_{(p,q)} \simeq_{n,\alpha,\beta,p} \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)},$$

where $\|[b, I_\alpha]\|_{(p,q)}$ is the best possible constant C in (6.61).

Remark 6.62. Let $1 \le p < \infty$ and consider the capacitary Lebesgue space

$$L^p(\mathcal{H}^\beta_\infty) := \left\{ f : \ f \text{ is } \mathcal{H}^\beta_\infty\text{-quasicontinuous and } \|f\|_{L^p(\mathcal{H}^\beta_\infty)} < \infty \right\},$$

where

$$\|f\|_{L^p(\mathcal{H}^\beta_\infty)} := \left(\int_{\mathbb{R}^n} |f|^p \, d\mathcal{H}^\beta_\infty\right)^{1/p}.$$

In the implication $(i) \Rightarrow (ii)$ in Theorem 6.61, the commutator estimate is proved for the dense subspace $L^p\left(\mathcal{H}^\beta_\infty\right) \cap C_c(\mathbb{R}^n)$. More precisely, there exists $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} |[b, I_\alpha]f|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} \leq C\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f|^p \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}, \tag{6.62}$$

for every $f \in L^p\left(\mathcal{H}^\beta_\infty\right) \cap C_c(\mathbb{R}^n)$. Since this subspace is dense in $L^p\left(\mathcal{H}^\beta_\infty\right)$ (see **[Ada98]** and **[BCR25, p. 6]**), the inequality (6.62) extends to all $f \in L^p\left(\mathcal{H}^\beta_\infty\right)$.

Our main ingredient to prove the previous result is a pointwise estimate for the β-dimensional sharp maximal function of the commutator. This result can be viewed as an extension of Theorem 1.3 in **[CF03]** to the setting of the Choquet integral with respect to Hausdorff content.

Theorem 6.63. *Let $B(t) = t\log(e + t)$. Fix $0 < \beta \leq n$ and $0 < \alpha < \beta$. Let b be a measurable function in* $\mathrm{BMO}^\beta(\mathbb{R}^n)$ *and let f be a nonnegative measurable function. Then there exists a constant $C = C(\alpha, \beta, n) > 0$ such that, for every $s > 1$ and every $x \in \mathbb{R}^n$,*

$$\mathcal{M}^\#_\beta([b, I_\alpha]f)(x) \leq C\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(\mathcal{M}_{\mathcal{H}^\beta_\infty}((I_\alpha f)^s)(x)^{\frac{1}{s}} + \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x)\right). \tag{6.63}$$

Moreover, if $\beta \in (n - \alpha, n]$, then there exists $C = C(\alpha, \beta, n) > 0$ such that, for $\mathcal{H}^\beta_\infty$-a.e. $x \in \mathbb{R}^n$,

$$\mathcal{M}^\#_\beta([b, I_\alpha]f)(x) \leq C\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(I_\alpha f(x) + \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x)\right). \tag{6.64}$$

Remark 6.64. By adapting the proof of Theorem 6.63, we can avoid the use of Orlicz maximal functions at the expense of obtaining a weaker inequality (see Remark 6.65). Specifically, for every $s, t > 1$, there exists a constant $C = C(\alpha, \beta, n, s, t) > 0$ such that for all $x \in \mathbb{R}^n$,

$$\mathcal{M}^\#_\beta([b, I_\alpha]f)(x) \leq C\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(\mathcal{M}_{\mathcal{H}^\beta_\infty}((I_\alpha f)^s)(x)^{\frac{1}{s}} + \mathcal{M}_{\alpha t,\mathcal{H}^\beta_\infty}(f^t)(x)^{\frac{1}{t}}\right).$$

Proof of Theorem 6.63. By homogeneity, it suffices to establish (6.63) for $x = 0$. Fix a cube Q centered at the origin. We must show that there exists a constant $c \in \mathbb{R}$ such that

$$\frac{1}{\ell(Q)^\beta} \int_Q |[b, I_\alpha]f - c| d\mathcal{H}^\beta_\infty \leq C\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(\mathcal{M}_{\mathcal{H}^\beta_\infty}((I_\alpha f)^s)(0)^{\frac{1}{s}} + \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(0)\right),$$

for all $s > 1$.

We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{Q^*}$ and $Q^* = 2Q$. Since b_{Q^*} is constant, we observe that $[b, I_\alpha]f = [b - b_{Q^*}, I_\alpha]f$. Consequently, for a constant $c_Q \in \mathbb{R}$ to be chosen later, we have

$$\begin{aligned}
\frac{1}{\ell(Q)^\beta} \int_Q |[b, I_\alpha]f - c_Q| d\mathcal{H}^\beta_\infty =& \frac{1}{\ell(Q)^\beta} \int_Q |[b - b_{Q^*}, I_\alpha]f - c_Q| d\mathcal{H}^\beta_\infty \\
=& \frac{1}{\ell(Q)^\beta} \int_Q |(b(x) - b_{Q^*})I_\alpha f(x) - I_\alpha((b - b_{Q^*})f)(x) - c_Q| d\mathcal{H}^\beta_\infty \\
\leq& \frac{1}{\ell(Q)^\beta} \int_Q |(b(x) - b_{Q^*})I_\alpha f(x)| d\mathcal{H}^\beta_\infty \\
&+ \frac{1}{\ell(Q)^\beta} \int_Q |I_\alpha((b - b_{Q^*})f_1)(x)| d\mathcal{H}^\beta_\infty \\
&+ \frac{1}{\ell(Q)^\beta} \int_Q |I_\alpha((b - b_{Q^*})f_2)(x) + c_Q| d\mathcal{H}^\beta_\infty \\
=& I_1 + I_2 + I_3.
\end{aligned}$$

We estimate each term separately. Let $s > 1$. By applying Hölder's inequality (iii) from Theorem 6.5 and Theorem 6.16, we obtain

$$\begin{aligned}
I_1 \leq & 2\left(\frac{1}{\ell(Q)^\beta}\int_Q |b(x)-b_{Q^*}|^{s'}d\mathcal{H}_\infty^\beta\right)^{\frac{1}{s'}}\left(\frac{1}{\ell(Q)^\beta}\int_Q |I_\alpha f(x)|^s d\mathcal{H}_\infty^\beta\right)^{\frac{1}{s}}\\
\leq & 2^{1+\frac{\beta}{s'}}\left(\frac{1}{\ell(Q^*)^\beta}\int_{Q^*} |b(x)-b_{Q^*}|^{s'}d\mathcal{H}_\infty^\beta\right)^{\frac{1}{s'}}\left(\frac{1}{\ell(Q)^\beta}\int_Q |I_\alpha f(x)|^s d\mathcal{H}_\infty^\beta\right)^{\frac{1}{s}}\\
\leq & 2^{1+\frac{\beta}{s'}}Cs'\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}\left(\frac{1}{\ell(Q)^\beta}\int_Q |I_\alpha f(x)|^s d\mathcal{H}_\infty^\beta\right)^{\frac{1}{s}}\\
\leq & 2^{1+\frac{\beta}{s'}}Cs'\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}\mathcal{M}_{\mathcal{H}_\infty^\beta}((I_\alpha f)^s)(0)^{\frac{1}{s}}. \qquad (6.65)
\end{aligned}$$

To estimate I_2, we apply Lemma 6.35 to the function $(b-b_{Q^*})f_1$, followed by Hölder's inequality (6.58) with $B(t) = t\log(e+t)$:

$$\begin{aligned}
I_2 \leq & C\ell(Q^*)^\alpha\frac{1}{\ell(Q^*)^\beta}\int_{Q^*}|(b-b_{Q^*})f_1|d\mathcal{H}_\infty^\beta\\
= & C\ell(Q^*)^\alpha\frac{1}{\ell(Q^*)^\beta}\int_{Q^*}|(b-b_{Q^*})f|d\mathcal{H}_\infty^\beta\\
\leq & C_\beta\ell(Q^*)^\alpha\|f\|_{B,Q^*,\mathcal{H}_\infty^\beta}\|b-b_{Q^*}\|_{\bar{B},Q^*,\mathcal{H}_\infty^\beta}\\
\leq & C_\beta\mathcal{M}_{\alpha,B,\mathcal{H}_\infty^\beta}f(0)\|b-b_{Q^*}\|_{\bar{B},Q^*,\mathcal{H}_\infty^\beta}\\
\leq & C_\beta\mathcal{M}_{\alpha,B,\mathcal{H}_\infty^\beta}f(0)\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)},
\end{aligned}$$

where in the last inequality we have used Remark 6.50, Theorem 6.18 and Remark 6.49.

We now choose $c_Q = -c$, where $c \in \mathbb{R}$ is the constant obtained by applying Lemma 6.36 to the function $(b-b_{Q^*})f_2$. Using Lemma 6.36, we have

$$\begin{aligned}
I_3 = & \frac{1}{\ell(Q)^\beta}\int_Q |I_\alpha((b-b_{Q^*})f_2)(x)-c|d\mathcal{H}_\infty^\beta\\
\leq & C\sum_{k=0}^\infty 2^{-k}(2^{k+1}\ell(Q))^\alpha\frac{1}{(2^{k+1}\ell(Q))^\beta}\int_{2^{k+1}Q}|(b-b_{Q^*})f_2|d\mathcal{H}_\infty^\beta\\
= & C\sum_{k=0}^\infty 2^{-k}(2^{k+1}\ell(Q))^\alpha\frac{1}{(2^{k+1}\ell(Q))^\beta}\int_{2^{k+1}Q}|(b-b_{2^{k+1}Q})f_2+(b_{2^{k+1}Q}-b_{Q^*})f_2|d\mathcal{H}_\infty^\beta\\
\leq & C_\beta\sum_{k=0}^\infty 2^{-k}(2^{k+1}\ell(Q))^\alpha\frac{1}{(2^{k+1}\ell(Q))^\beta}\int_{2^{k+1}Q}|b-b_{2^{k+1}Q}||f|d\mathcal{H}_\infty^\beta\\
& + C_\beta\sum_{k=0}^\infty 2^{-k}|b_{2^{k+1}Q}-b_{Q^*}|(2^{k+1}\ell(Q))^\alpha\frac{1}{(2^{k+1}\ell(Q))^\beta}\int_{2^{k+1}Q}|f|d\mathcal{H}_\infty^\beta\\
= & I_{31}+I_{32}.
\end{aligned}$$

To estimate I_{31}, we repeat the argument used for I_2:

$$I_{31} \leq C_\beta\sum_{k=0}^\infty 2^{-k}(2^{k+1}\ell(Q))^\alpha\|f\|_{B,2^{k+1}Q,\mathcal{H}_\infty^\beta}\|b-b_{2^{k+1}Q}\|_{\bar{B},2^{k+1}Q,\mathcal{H}_\infty^\beta}$$

$$\begin{aligned}
&\leq C_\beta \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(0) \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \sum_{k=0}^{\infty} 2^{-k} \\
&= C_\beta \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(0) \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}.
\end{aligned}$$

To estimate I_{32}, we invoke Lemma 6.17 and Remark 6.53:

$$\begin{aligned}
I_{32} &= C_\beta \sum_{k=0}^{\infty} 2^{-k} |b_{2^k Q^*} - b_{Q^*}| (2^{k+1}\ell(Q))^\alpha \frac{1}{(2^{k+1}\ell(Q))^\beta} \int_{2^{k+1}Q} |f| d\mathcal{H}^\beta_\infty \\
&\leq C_\beta \sum_{k=0}^{\infty} 2^{-k} k \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} (2^{k+1}\ell(Q))^\alpha \frac{1}{(2^{k+1}\ell(Q))^\beta} \int_{2^{k+1}Q} |f| d\mathcal{H}^\beta_\infty \\
&\leq C_\beta \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \mathcal{M}_{\alpha,\mathcal{H}^\beta_\infty} f(0) \sum_{k=0}^{\infty} 2^{-k} k \\
&\leq C_\beta \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(0).
\end{aligned}$$

This concludes the proof of the first statement. Now assume that $\beta \in (n-\alpha, n]$. By Lemma 6.37, we have $I_\alpha f \in A_1^\beta$, and hence $\mathcal{M}_{\mathcal{H}^\beta_\infty}(I_\alpha f)(x) \leq C I_\alpha f(x)$ for $\mathcal{H}^\beta_\infty$-a.e. $x \in \mathbb{R}^n$. Fix such a point and, by translation invariance, assume that it is 0. Returning to (6.65), we choose $s > 1$ to be the exponent provided by the reverse Hölder inequality (Theorem 6.38). We then obtain

$$\begin{aligned}
I_1 &\leq 2^{1+\frac{\beta}{s'}} C s' \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(\frac{1}{\ell(Q)^\beta} \int_Q |I_\alpha f(x)|^s d\mathcal{H}^\beta_\infty \right)^{\frac{1}{s}} \\
&\leq 2^{1+\frac{\beta}{s'}} C s' \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \frac{1}{\ell(Q)^\beta} \int_Q |I_\alpha f(x)| d\mathcal{H}^\beta_\infty \\
&\leq 2^{1+\frac{\beta}{s'}} C s' \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} I_\alpha f(0),
\end{aligned}$$

where in the last inequality we have used Lemma 6.37 again. This establishes (6.64). □

Remark 6.65. We indicate how the argument in the proof of Theorem 6.63 also yields the pointwise estimate stated in Remark 6.64, without appealing to the Orlicz fractional maximal function. Using the notation of the proof of Theorem 6.63, it suffices to revisit the terms I_2 and I_3. Fix $t > 1$. Replacing the use of Lemma 6.51 by the Hölder inequality stated in Theorem 6.5 (iii) with exponent t, and combining it with Theorem 6.16, we obtain

$$I_2 + I_3 \lesssim \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \mathcal{M}_{\alpha t,\mathcal{H}^\beta_\infty}(f^t)(x)^{1/t}.$$

Together with the estimate for I_1 already established in the proof of Theorem 6.63, this yields that for every $s, t > 1$,

$$\mathcal{M}^\#_\beta([b, I_\alpha]f)(x) \lesssim \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(\mathcal{M}_{\mathcal{H}^\beta_\infty}((I_\alpha f)^s)(x)^{\frac{1}{s}} + \mathcal{M}_{\alpha t,\mathcal{H}^\beta_\infty}(f^t)(x)^{\frac{1}{t}} \right),$$

which is precisely the desired bound.

Remark 6.66. The previous proof can be extended naturally to iterated commutators. For $m \in \mathbb{N}$ and b locally integrable, we define

$$(I_\alpha)^1_b f := [b, I_\alpha] f, \qquad (I_\alpha)^m_b f := \left[b, (I_\alpha)^{m-1}_b\right] f, \quad m \geq 2.$$

Equivalently,

$$(I_\alpha)_b^m f(x) = \int_{\mathbb{R}^n} \frac{(b(x)-b(y))^m}{|x-y|^{n-\alpha}}\, f(y)\, dy.$$

In particular, $(I_\alpha)_b^m$ is invariant under the addition of constants to b. The proof of Theorem 6.63 adapts to $(I_\alpha)_b^m$ by using the binomial expansion

$$(I_\alpha)_b^m f(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j\, b(x)^{m-j}\, I_\alpha\big(b^j f\big)(x),$$

applied with $b_0 := b - b_{Q^*}$, and treating each summand as in the case $m = 1$ after splitting $f = f_1 + f_2$ where $f_1 = f\chi_{Q^*}$. This yields the analogue of (6.63) with the natural dependence $\|b\|^m_{\mathrm{BMO}^\beta(\mathbb{R}^n)}$ and with the Young function B replaced by $B_m(t) := t(\log(e+t))^m$. Consequently, the corresponding analogue of Theorem 6.67 follows by the same argument. We do not pursue these extensions here.

We now present the proof of Theorem 6.61. For clarity, we divide the characterization into two separate results: first, we show that the condition $b \in \mathrm{BMO}^\beta$ is sufficient for the boundedness of the commutator $[b, I_\alpha]$, and later we prove that this condition is also necessary.

Theorem 6.67. *Let $0 < \beta \le n$, $0 < \alpha < \beta$ and let b be a measurable function. For each $1 < p < \frac{\beta}{\alpha}$, we define the exponent q by*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\beta}.$$

If $b \in \mathrm{BMO}^\beta(\mathbb{R}^n)$, then there exists $C = C(\alpha, \beta, n, p) > 0$ such that

$$\left(\int_{\mathbb{R}^n} |[b, I_\alpha]f|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} \le C\|b\|_{\mathrm{BMO}^\beta} \left(\int_{\mathbb{R}^n} |f|^p \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}, \tag{6.66}$$

for every bounded measurable function f with compact support.

Proof. Let $b \in \mathrm{BMO}^\beta(\mathbb{R}^n)$ and let f be a bounded measurable function with compact support. We may assume without loss of generality that $\|b\|_{\mathrm{BMO}^\beta} \neq 0$. For $k > 0$, we define the truncated function b_k as in Lemma 6.20, hence

$$\|b_k\|_{\mathrm{BMO}^\beta} \le C_\beta \|b\|_{\mathrm{BMO}^\beta}.$$

Fix some $p_0 \in (1, p)$, and define q_0 by the relation $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{\beta}$. Since f has compact support and b_k is bounded, $\|b_k f\|_{L^{p_0}(\mathcal{H}^\beta_\infty)} < \infty$ and by Theorem 6.33 we have $\|I_\alpha(b_k f)\|_{L^{q_0}(\mathcal{H}^\beta_\infty)} < \infty$. Also, since $\|f\|_{L^{p_0}(\mathcal{H}^\beta_\infty)} < \infty$, again by Theorem 6.33 we get $\|I_\alpha f\|_{L^{q_0}(\mathcal{H}^\beta_\infty)} < \infty$ and hence $\|b_k I_\alpha f\|_{L^{q_0}(\mathcal{H}^\beta_\infty)} < \infty$. Therefore, $[b_k, I_\alpha]f$ is well defined and we have

$$\|[b_k, I_\alpha]f\|_{L^{q_0}(\mathcal{H}^\beta_\infty)} = \|b_k I_\alpha f - I_\alpha(b_k f)\|_{L^{q_0}(\mathcal{H}^\beta_\infty)} < \infty.$$

In particular, the weak-type estimate (6.17) needed to apply Theorem 6.22 is satisfied by $[b_k, I_\alpha]f$.

Using the Lebesgue differentiation theorem **[CC25, Theorem 2.9]** and Theorem 6.22, we deduce that there exists a constant $C > 0$ depending only on q, n, β such that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |[b_k, I_\alpha]f|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} &\le \left(\int_{\mathbb{R}^n} \left|\mathcal{M}_{\mathcal{H}^\beta_\infty}([b_k, I_\alpha]f)\right|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} \\ &\le C\left(\int_{\mathbb{R}^n} \left|\mathcal{M}^\#_\beta([b_k, I_\alpha]f)\right|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}}.\end{aligned}$$

Now, by using the pointwise inequality given in Theorem 6.63 (concretely the inequality given in Remark 6.64) with $1 < s, t < q$, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left|\mathcal{M}^{\#}_{\beta}([b_k, I_\alpha]f)\right|^q d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} &\leq C\|b_k\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}\left|\mathcal{M}_{\mathcal{H}^\beta_\infty}((I_\alpha f)^s)(x)\right|^{\frac{q}{s}} d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}}\\ &\quad + C\|b_k\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}\left|\mathcal{M}_{\alpha t,\mathcal{H}^\beta_\infty}(f^t)(x)\right|^{\frac{q}{t}} d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}}\\ &\leq C\|b\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}\left|\mathcal{M}_{\mathcal{H}^\beta_\infty}((I_\alpha f)^s)(x)\right|^{\frac{q}{s}} d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}}\\ &\quad + C\|b\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}\left|\mathcal{M}_{\alpha t,\mathcal{H}^\beta_\infty}(f^t)(x)\right|^{\frac{q}{t}} d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}}\\ &= I_1 + I_2.\end{aligned}$$

We will estimate each term separately. Firstly, (choosing s such that $\frac{q}{s} > 1$), using the boundedness of $\mathcal{M}_{\mathcal{H}^\beta_\infty}$ proved in **[COS24, Theorem 1.2]**, we have

$$\begin{aligned}I_1 &\leq C\|b\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}|I_\alpha f|^q\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}}\\ &\leq C\|b\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}|f|^p\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}},\end{aligned}$$

where in the last inequality we have used Theorem 6.33. We estimate the second term using Theorem 6.34, where we use the relation

$$\frac{t}{p} - \frac{t}{q} = \frac{t\alpha}{\beta},$$

therefore

$$\begin{aligned}I_2 =& C\|b_k\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}\left|\mathcal{M}_{\alpha t,\mathcal{H}^\beta_\infty}(f^t)(x)\right|^{\frac{q}{t}} d\mathcal{H}^\beta_\infty\right)^{\frac{t}{q}\frac{1}{t}}\\ \leq& C\|b\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}|f|^{t\frac{p}{t}}\, d\mathcal{H}^\beta_\infty\right)^{\frac{t}{p}\frac{1}{t}}\\ =& C\|b\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}|f|^p\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}.\end{aligned}$$

Hence, we have proved

$$\left(\int_{\mathbb{R}^n}|[b_k, I_\alpha]f|^q\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} \leq C\|b\|_{\mathrm{BMO}^\beta}\left(\int_{\mathbb{R}^n}|f|^p\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}},$$

for every $k > 0$ with constant C independent of k.

To conclude the proof for the general symbol b we take the limit as $k \to \infty$. Since f is bounded and has compact support, let us denote $K = \mathrm{supp} f$. By the monotone convergence theorem, we have

$$\left(\int_{\mathbb{R}^n}|b_k(x)f(x) - b(x)f(x)|^s\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{s}} = \left(\int_K |b_k(x) - b(x)|^s|f(x)|^s\, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{s}} \to 0,$$

as $k \to \infty$. Applying Theorem 6.33,

$$\left(\int_{\mathbb{R}^n} |I_\alpha(b_k f)(x) - I_\alpha(bf)(x)|^r \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{r}} = \left(\int_{\mathbb{R}^n} |I_\alpha(b_k f - bf)(x)|^r \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{r}}$$
$$\leq C \left(\int_{\mathbb{R}^n} |b_k(x)f(x) - b(x)f(x)|^s \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{s}} \to 0,$$

as $k \to \infty$, where r is the corresponding exponent given by Theorem 6.33. This norm convergence implies that there exists a subsequence $\{b_{k_j}\}_{j\in\mathbb{N}}$ such that $b_{k_j} \to b$ and $I_\alpha(b_{k_j} f) \to I_\alpha(bf)$ $\mathcal{H}^\beta_\infty$-a.e. (see, for example, **[PS23, Proposition 2.5]**). Consequently, $[b_{k_j}, I_\alpha]f(x) \to [b, I_\alpha]f(x)$ for $\mathcal{H}^\beta_\infty$-a.e. $x \in \mathbb{R}^n$. Therefore, by Fatou's lemma, we conclude

$$\left(\int_{\mathbb{R}^n} |[b, I_\alpha]f|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^n} \lim_{j\to\infty} \left|[b_{k_j}, I_\alpha]f\right|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}}$$
$$\leq \liminf_{j\to\infty} \left(\int_{\mathbb{R}^n} |[b_{k_j}, I_\alpha]f|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}}$$
$$\leq C \liminf_{j\to\infty} \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f|^p \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}$$
$$= C \|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f|^p \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}.$$

□

We now prove the converse implication, namely that the boundedness of the commutator implies that the symbol b belongs to BMO^β. This establishes the necessity of the condition and completes the characterization.

Theorem 6.68. *Let $0 < \beta \leq n$, $0 < \alpha < \beta$ and let $b \in L^1_{loc}(\mathbb{R}^n)$. For each $1 < p < \frac{\beta}{\alpha}$, we define the exponent q by*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\beta}.$$

If the commutator $[b, I_\alpha]$ satisfies the inequality

$$\left(\int_{\mathbb{R}^n} |[b, I_\alpha]f|^q \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f|^p \, d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}, \tag{6.67}$$

for some $C > 0$ and for every bounded measurable function f with compact support, then $b \in \mathrm{BMO}^\beta(\mathbb{R}^n)$. Specifically, if we denote by $\|[b, I_\alpha]\|_{(p,q)}$ the best possible constant in the inequality (6.67)*, there exists a constant $C = C(\alpha, \beta, n) > 0$ such that $\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \leq C\|[b, I_\alpha]\|_{(p,q)}$.*

The proof follows the lines of the proof of Lemma 5.1 in **[HRS16]**, with slight modifications to adapt it to our nonlinear context. We also refer to **[Jan78]** and **[Cha16]**, where similar proofs are done for Calderón–Zygmund singular integrals in the scalar and bilinear case, respectively.

Proof. Fix a cube Q. Let us consider the cube P with $\ell(P) = 4\ell(Q)$, with the same bottom left corner. Let P_R be the upper right half of P. We observe that $\ell(P_R) = 2\ell(Q)$. Given $x \in Q$ and $y \in P_R$, we have the following trivial estimates,

$$\frac{|x-y|}{2\sqrt{n}\ell(P)} \geq \frac{\sqrt{n}\ell(Q)}{2\sqrt{n}\ell(P)} = \frac{1}{8},$$

and,

$$\frac{|x-y|}{2\sqrt{n}\ell(P)} \leq \frac{\sqrt{n}\ell(P)}{2\sqrt{n}\ell(P)} = \frac{1}{2}.$$

There exists a function $K(x)$, that is smooth on $[-1,1]^n$, has a smooth periodic extension to $\mathbb{R}^n$, and is equal to $|x|^{n-\alpha}$ for $\frac{1}{8} \leq |x| \leq \frac{1}{2}$. Therefore, for $x \in Q$ and $y \in P_R$, we have

$$\left(\frac{|x-y|}{2\sqrt{n}\ell(P)}\right)^{n-\alpha} = K\left(\frac{x-y}{2\sqrt{n}\ell(P)}\right).$$

We can express the function K as an absolutely convergent Fourier series.

Now, we define $c_Q = \frac{1}{|P_R|}\int_{P_R} b(y)dy$ and $\sigma(x) = \operatorname{sgn}(b(x) - c_Q)$. Then, we have

$$\begin{aligned}
\int_Q |b - c_Q| d\mathcal{H}^\beta_\infty =& \int_Q \left| b(x) - \frac{1}{|P_R|}\int_{P_R} b(y)dy \right| d\mathcal{H}^\beta_\infty(x) \\
=& \frac{1}{\ell(P_R)^n}\int_Q \left| \int_{P_R} (b(x) - b(y))\sigma(x)dy \right| d\mathcal{H}^\beta_\infty(x) \\
=& \frac{1}{\ell(P_R)^n}\int_Q \left| \int_{P_R} \frac{b(x)-b(y)}{\left(\frac{|x-y|}{2\sqrt{n}\ell(P)}\right)^{n-\alpha}} \left(\frac{|x-y|}{2\sqrt{n}\ell(P)}\right)^{n-\alpha} \sigma(x)dy \right| d\mathcal{H}^\beta_\infty(x) \\
=& C_{n,\alpha}\frac{1}{\ell(P)^\alpha}\int_Q \left| \int_{P_R} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} K\left(\frac{x-y}{2\sqrt{n}\ell(P)}\right)\sigma(x)dy \right| d\mathcal{H}^\beta_\infty(x) \\
=& I_1
\end{aligned}$$

We expand K as an absolutely convergent Fourier series,

$$K\left(\frac{x-y}{2\sqrt{n}\ell(P)}\right) = \sum_k a_k e^{ik\frac{x}{2\sqrt{n}\ell(P)}} e^{-ik\frac{y}{2\sqrt{n}\ell(P)}},$$

where we remark that $\sum_k |a_k| = C_{\alpha,n} < \infty$. Using the absolute convergence of the Fourier series of K, the monotonicity of the Choquet integral, and (6.5), we obtain

$$\begin{aligned}
I_1 \leq& C_{n,\alpha,\beta}\frac{1}{\ell(P)^\alpha}\sum_k |a_k| \int_Q \left| \int_{P_R} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} e^{-ik\frac{y}{2\sqrt{n}\ell(P)}} e^{ik\frac{x}{2\sqrt{n}\ell(P)}}\sigma(x)dy \right| d\mathcal{H}^\beta_\infty(x) \\
=& C_{n,\alpha,\beta}\frac{1}{\ell(P)^\alpha}\sum_k |a_k| \int_Q \left| \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} e^{-ik\frac{y}{2\sqrt{n}\ell(P)}} \chi_{P_R}(y)dy\, e^{ik\frac{x}{2\sqrt{n}\ell(P)}}\sigma(x) \right| d\mathcal{H}^\beta_\infty(x) \\
=& C_{n,\alpha,\beta}\frac{1}{\ell(P)^\alpha}\sum_k |a_k| \int_Q \left| [b, I_\alpha] f_k(x) e^{ik\frac{x}{2\sqrt{n}\ell(P)}}\sigma(x) \right| d\mathcal{H}^\beta_\infty(x) \\
=& C_{n,\alpha,\beta}\frac{1}{\ell(P)^\alpha}\sum_k |a_k| \int_Q |[b, I_\alpha] f_k(x)|\, d\mathcal{H}^\beta_\infty(x) \\
=& I_2,
\end{aligned}$$

where $f_k(y) = e^{-ik\frac{y}{2\sqrt{n}\ell(P)}}\chi_{P_R}(y)$. We can estimate each integral using Hölder's inequality and (6.67),

$$\begin{aligned}\int_Q |[b,I_\alpha]f_k(x)|\,d\mathcal{H}^\beta_\infty(x) &= \int_{\mathbb{R}^n} |[b,I_\alpha]f_k(x)|\,\chi_Q(x)d\mathcal{H}^\beta_\infty(x)\\ &\leq C_\beta\left(\int_Q |[b,I_\alpha]f_k(x)|^q\,d\mathcal{H}^\beta_\infty(x)\right)^{\frac{1}{q}}\ell(Q)^{\frac{\beta}{q'}}\\ &\leq C_\beta\|[b,I_\alpha]\|_{(p,q)}\left(\int_{\mathbb{R}^n}|f_k|^p\,d\mathcal{H}^\beta_\infty\right)^{\frac{1}{p}}\ell(Q)^{\frac{\beta}{q'}}\\ &\leq C_\beta\|[b,I_\alpha]\|_{(p,q)}\ell(P_R)^{\frac{\beta}{p}}\ell(Q)^{\frac{\beta}{q'}}\\ &\leq C_{n,\beta}\|[b,I_\alpha]\|_{(p,q)}\ell(Q)^{\frac{\beta}{p}+\frac{\beta}{q'}}\\ &= C_{n,\beta}\|[b,I_\alpha]\|_{(p,q)}\ell(Q)^{\alpha+\beta},\end{aligned}$$

where we have used the relation $\frac{1}{q'}+\frac{1}{p}=\frac{\alpha}{\beta}+1$. Then, we have

$$\begin{aligned}I_2 &\leq C_{n,\alpha,\beta}\frac{1}{\ell(P)^\alpha}\sum_k |a_k|\|[b,I_\alpha]\|_{(p,q)}\ell(Q)^{\alpha+\beta}\\ &\leq C_{n,\alpha,\beta}\frac{1}{\ell(Q)^\alpha}\|[b,I_\alpha]\|_{(p,q)}\ell(Q)^{\alpha+\beta}\sum_k|a_k|\\ &= C_{n,\alpha,\beta}\|[b,I_\alpha]\|_{(p,q)}\ell(Q)^\beta.\end{aligned}$$

Therefore, we have proved that for each cube Q there exist $c_Q\in\mathbb{R}$ such that

$$\frac{1}{\ell(Q)^\beta}\int_Q |b-c_Q|d\mathcal{H}^\beta_\infty \leq C_{n,\alpha,\beta}\|[b,I_\alpha]\|_{(p,q)},$$

for all cubes Q, and this implies $\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}\leq C_{n,\alpha,\beta}\|[b,I_\alpha]\|_{(p,q)}$. □

This completes the proof of Theorem 6.61, upon combining Theorems 6.67 and 6.68.

6.8 Endpoint modular weak-type inequality

In the classical Euclidean setting, the endpoint estimates for the commutator of the Riesz potential, $[b,I_\alpha]$, were established by Cruz-Uribe and Fiorenza **[CF03]** (see also **[DLZ01]**) and by Adams and Xiao **[AX12]** for the critical cases $p=1$ and $p=\frac{n}{\alpha}$, respectively. In particular, for the lower endpoint $p=1$, Cruz-Uribe and Fiorenza obtained a modular weak-type estimate of the form

$$|\{x\in\mathbb{R}^n : |[b,I_\alpha]f(x)|>t\}| \leq C\,\Psi\left(\int_{\mathbb{R}^n} B\left(\|b\|_{\mathrm{BMO}}\frac{|f(x)|}{t}\right)dx\right),$$

for all $t>0$, where $B(t)=t\log(e+t)$ and $\Psi(t)=[t\log(e+t^{\alpha/n})]^{n/(n-\alpha)}$. We extend this classical inequality to the capacitary framework by replacing the underlying Lebesgue measure with the Hausdorff content $\mathcal{H}^\beta_\infty$ and considering symbols in $\mathrm{BMO}^\beta(\mathbb{R}^n)$. A key ingredient in our proof is the boundedness of the capacitary fractional maximal operator $\mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty}$, stated in Theorem 6.59.

We now state the main result of this section.

Theorem 6.69. *Let $0 < \beta \le n$, $0 < \alpha < \beta$, and assume that $\beta \in (n-\alpha, n]$. Let b be a measurable function in $\mathrm{BMO}^\beta(\mathbb{R}^n)$. Define $B(t) = t\log(e+t)$ and $\Psi(t) = [t\log(e+t^{\alpha/\beta})]^{\beta/(\beta-\alpha)}$. Then, there exists a constant $C = C(\alpha,\beta,n) > 0$ such that for every bounded measurable function f with compact support, we have*

$$\mathcal{H}^\beta_\infty\big(\{x\in\mathbb{R}^n : |[b,I_\alpha]f(x)| > t\}\big) \le C\,\Psi\left(\int_{\mathbb{R}^n} B\left(\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)}\frac{|f(x)|}{t}\right) d\mathcal{H}^\beta_\infty\right), \tag{6.68}$$

for all $t > 0$.

Remark 6.70. The restriction $\beta \in (n-\alpha, n]$ is the same dimensional constraint discussed in Remark 6.41. It reflects the lack of Choquet integrability of the kernel $|x|^{\alpha-n}$ with respect to $\mathcal{H}^\beta_\infty$ when $\beta \le n-\alpha$.

Remark 6.71. When $\beta = n$, Theorem 6.69 reduces to the endpoint modular estimate of Cruz-Uribe and Fiorenza, who proved that the corresponding inequality is sharp: if it holds with Ψ replaced by an increasing function Ψ_0, then necessarily $\Psi(t/\gamma) \le K\,\Psi_0(t)$ for all $t > 0$ and suitable constants $\gamma, K > 0$. In particular, the logarithmic growth encoded in Ψ cannot be improved even in the Lebesgue case.

To prove the modular weak-type inequality stated in Theorem 6.69, we require the following modification of our Fefferman–Stein inequality.

Lemma 6.72. *Let $\varphi : (0,\infty) \longrightarrow (0,\infty)$ be a doubling function, that is, there exists $C_\varphi > 0$ such that $\varphi(2t) \le C_\varphi\varphi(t)$ for every $t > 0$. Then, there exists a constant $C = C(\beta,\varphi) > 0$ such that*

$$\sup_{\lambda>0}\varphi(\lambda)\mathcal{H}^\beta_\infty(\{x : \mathcal{M}_{\mathcal{H}^\beta_\infty,\mathcal{Q}_0}f(x) > \lambda\}) \le C\sup_{\lambda>0}\varphi(\lambda)\mathcal{H}^\beta_\infty(\{x : \mathcal{M}^\#_\beta f(x) > \lambda\}), \tag{6.69}$$

for all functions f such that the left-hand side is finite.

The proof adapts the arguments of **[Pér95, Lemma 2.1]**, relying on the good-lambda inequality established in Theorem 6.24.

Proof. In view of the equivalence (6.4), it suffices to establish (6.69) for the dyadic Hausdorff content $\mathcal{H}^{\beta,\mathcal{Q}_0}_\infty$. By the good-lambda inequality proved in Theorem 6.24, there exists a constant $C_\beta > 0$ such that

$$\begin{aligned}\mathcal{H}^{\beta,\mathcal{Q}_0}_\infty(\{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,\mathcal{Q}_0}_\infty}f(x) > \lambda\}) \le{}& \mathcal{H}^{\beta,\mathcal{Q}_0}_\infty(\{x\in\mathbb{R}^n : \mathcal{M}^\#_\beta f(x) > \varepsilon\lambda\})\\ &+ \varepsilon\,C_\beta\mathcal{H}^{\beta,\mathcal{Q}_0}_\infty(\{x\in\mathbb{R}^n : \mathcal{M}_{\mathcal{H}^{\beta,\mathcal{Q}_0}_\infty}f(x) > c_\beta\lambda\}),\end{aligned}$$

for all $\lambda > 0$ and $\varepsilon > 0$, where $c_\beta = \frac{1}{2^{\beta+2}}$. Using the doubling property of φ, there exists a constant $C_1 = C_1(\beta,\varphi) > 0$ such that $\varphi(\lambda) = \varphi(2^{\beta+2}c_\beta\lambda) \le C_1\varphi(c_\beta\lambda)$ for all $\lambda > 0$. We choose $\varepsilon > 0$

sufficiently small so that $\varepsilon C_\beta C_1 = \frac{1}{2}$. Invoking the doubling property again, there exists a constant $C_2 = C_2(\beta, \varphi) > 0$ such that $\varphi(\lambda) = \varphi(\frac{1}{\varepsilon}\varepsilon\lambda) \leq C_2\varphi(\varepsilon\lambda)$. Consequently, we have

$$\begin{aligned}
\varphi(\lambda)\mathcal{H}^{\beta,Q_0}_\infty(\{x : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > \lambda\}) \leq & C_2\varphi(\varepsilon\lambda)\mathcal{H}^{\beta,Q_0}_\infty(\{x : \mathcal{M}^\#_\beta f(x) > \varepsilon\lambda\}) \\
& + \frac{1}{2}\varphi(c_\beta\lambda)\mathcal{H}^{\beta,Q_0}_\infty(\{x : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > c_\beta\lambda\}) \\
\leq & C_2 \sup_{\lambda>0} \varphi(\varepsilon\lambda)\mathcal{H}^{\beta,Q_0}_\infty(\{x : \mathcal{M}^\#_\beta f(x) > \varepsilon\lambda\}) \\
& + \frac{1}{2}\sup_{\lambda>0} \varphi(c_\beta\lambda)\mathcal{H}^{\beta,Q_0}_\infty(\{x : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > c_\beta\lambda\}) \\
= & C_2 \sup_{\lambda>0} \varphi(\lambda)\mathcal{H}^{\beta,Q_0}_\infty(\{x : \mathcal{M}^\#_\beta f(x) > \lambda\}) \\
& + \frac{1}{2}\sup_{\lambda>0} \varphi(\lambda)\mathcal{H}^{\beta,Q_0}_\infty(\{x : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty} f(x) > \lambda\}).
\end{aligned}$$

Taking the supremum over $\lambda > 0$ on the left-hand side and assuming it is finite, we may absorb the second term on the right-hand side to conclude the proof of (6.69) with $C = 2C_2$. □

By combining the pointwise estimate (6.64) and the capacitary Fefferman–Stein inequality (6.69) with the boundedness properties of the Riesz potential and the fractional maximal operator, we obtain the desired endpoint control.

Proof of Theorem 6.69. The proof follows the lines of the proof of Theorem 1.2 in **[CF03]**. We refer to that paper for more details. We may assume that $\|b\|_{\mathrm{BMO}^\beta(\mathbb{R}^n)} \neq 0$ and that f is non-negative. By homogeneity it is enough to prove (6.68) when $t = 1$. We have

$$\begin{aligned}
\mathcal{H}^\beta_\infty(\{x : |[b, I_\alpha]f(x)| > 1\}) = & \Psi(B(1))\frac{1}{\Psi(B(1))}\mathcal{H}^\beta_\infty(\{x : |[b, I_\alpha]f(x)| > 1\}) \\
\leq & \Psi(B(1)) \sup_{t>0} \frac{1}{\Psi(B(1/t))}\mathcal{H}^\beta_\infty(\{x : |[b, I_\alpha]f(x)| > t\}) \\
\leq & \Psi(B(1)) \sup_{t>0} \frac{1}{\Psi(B(1/t))}\mathcal{H}^\beta_\infty(\{x : \mathcal{M}_{\mathcal{H}^{\beta,Q_0}_\infty}([b, I_\alpha]f)(x) > t\}).
\end{aligned}$$

Since $\varphi(t) = \frac{1}{\Psi(B(1/t))}$ is doubling **[CF03, p. 123]**, we can apply the modular Fefferman–Stein inequality (6.69). Thus, by (6.69) and Theorem 6.63, we have

$$\begin{aligned}
\mathcal{H}^\beta_\infty(\{x : |[b, I_\alpha]f(x)| > 1\}) \leq & C \sup_{t>0} \frac{1}{\Psi(B(1/t))}\mathcal{H}^\beta_\infty(\{x : \mathcal{M}^\#_\beta([b, I_\alpha]f)(x) > t\}) \\
\leq & C \sup_{t>0} \frac{1}{\Psi(B(1/t))}\mathcal{H}^\beta_\infty\left(\left\{x : I_\alpha f(x) > \frac{t}{C\|b\|_{\mathrm{BMO}^\beta}}\right\}\right) \\
& + C \sup_{t>0} \frac{1}{\Psi(B(1/t))}\mathcal{H}^\beta_\infty\left(\left\{x : \mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty} f(x) > \frac{t}{C\|b\|_{\mathrm{BMO}^\beta}}\right\}\right) \\
= & I_1 + I_2.
\end{aligned}$$

We need to use the weak-type estimate for the Riesz potential in the capacitary context to estimate I_1. More precisely, we will use Theorem 6.33 in the case $p = 1$,

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^{\frac{\beta}{\beta-\alpha}} d\mathcal{H}^\beta_\infty\right)^{\frac{\beta-\alpha}{\beta}} \leq C \int_{\mathbb{R}^n} |f(x)| d\mathcal{H}^\beta_\infty.$$

Note that for $0 < \beta < n$ the previous inequality is not the endpoint, $p = \frac{\beta}{n}$. However, for each $0 < \beta \leq n$ we have the following weak-type inequality,

$$\mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \leq C \left(\frac{1}{t}\left(\int_{\mathbb{R}^n} |f(x)| d\mathcal{H}^\beta_\infty\right)\right)^{\frac{\beta}{\beta-\alpha}},$$

for every $t > 0$. Using the previous estimate, we have

$$\begin{aligned}
I_1 \leq & C \sup_{t>0} \frac{1}{\Psi(B(1/t))} \left(\frac{\|b\|_{\mathrm{BMO}^\beta}}{t} \int_{\mathbb{R}^n} |f(x)| d\mathcal{H}^\beta_\infty\right)^{\frac{\beta}{\beta-\alpha}} \\
= & C \sup_{t>0} \frac{1}{\Psi(B(1/t))} \frac{1}{t^{\beta/(\beta-\alpha)}} \left(\int_{\mathbb{R}^n} \left(\|b\|_{\mathrm{BMO}^\beta} |f(x)|\right) d\mathcal{H}^\beta_\infty\right)^{\frac{\beta}{\beta-\alpha}} \\
\leq & C \sup_{t>0} \frac{1}{\Psi(B(1/t))} \frac{1}{t^{\beta/(\beta-\alpha)}} \left(\int_{\mathbb{R}^n} B\left(\|b\|_{\mathrm{BMO}^\beta} |f(x)|\right) d\mathcal{H}^\beta_\infty\right)^{\frac{\beta}{\beta-\alpha}} \\
\leq & C \Psi\left(\int_{\mathbb{R}^n} B\left(\|b\|_{\mathrm{BMO}^\beta} |f(x)|\right) d\mathcal{H}^\beta_\infty\right)
\end{aligned}$$

since $t \leq B(t)$ and $t^{\beta/(\beta-\alpha)} \leq \Psi(t)$ for all $t > 0$, and in the last inequality we have used

$$\sup_{t>0} \frac{1}{\Psi(B(1/t))} \frac{1}{t^{\beta/(\beta-\alpha)}} \leq C,$$

we refer the reader to **[CF03, p. 124]** for more details.

To estimate I_2, we use the weak-type estimate for the capacitary Orlicz maximal function $\mathcal{M}_{\alpha,B,\mathcal{H}^\beta_\infty}$ established in Theorem 6.59, obtaining

$$\begin{aligned}
I_2 \leq & C \sup_{t>0} \frac{1}{\Psi(B(1/t))} \Psi\left(\int_{\mathbb{R}^n} B\left(\frac{\|b\|_{\mathrm{BMO}^\beta} |f(x)|}{t}\right) d\mathcal{H}^\beta_\infty\right) \\
\leq & C \sup_{t>0} \frac{1}{\Psi(B(1/t))} \Psi(B(1/t)) \Psi\left(\int_{\mathbb{R}^n} B\left(\|b\|_{\mathrm{BMO}^\beta} |f(x)|\right) d\mathcal{H}^\beta_\infty\right) \\
\leq & C \Psi\left(\int_{\mathbb{R}^n} B\left(\|b\|_{\mathrm{BMO}^\beta} |f(x)|\right) d\mathcal{H}^\beta_\infty\right),
\end{aligned}$$

where we have used that B and Ψ are submultiplicative functions. This concludes the proof. □

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