SELF-IMPROVING POINCARÉ-TYPE INEQUALITIES IN LORENTZ SPACES AND APPLICATIONS

Alejandro Claros
BCAM - Basque Center for Applied Mathematics
aclaros@bcamath.org

Abstract

We study local weighted Poincaré and Poincaré-Sobolev type inequalities with Lorentz quasi-norm $L^{p,q}$ on the left hand side, where $1 < p < \infty$ and $0 < q < \infty$. We obtain those inequalities as a consequence of a general self-improving property shared by functions satisfying the generalized Poincaré inequality

$$\frac{1}{|Q|}\int_Q |f - f_Q| dx \leq a(Q)$$

for all cubes $Q \subset \mathbb{R}^n$, where $f_Q = \frac{1}{|Q|}\int_Q f$ and a is some functional over cubes that satisfy the $SD_p^s(w)$ condition.

Preliminaries and known results

Weighted Lorentz spaces

- Let w be a weight function, i.e., a locally integrable nonnegative function. We will use the following notation for the weighted Lorentz average over a cube Q, given $1 \leq p < \infty$,

$$\|f\|_{L^{p,q}\left(Q,\frac{w(x)dx}{w(Q)}\right)} = \left(p\int_0^\infty t^q \left(\frac{w(\{x \in Q : |f(x)| > t\})}{w(Q)}\right)^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}}$$

whenever $q < \infty$, and

$$\|f\|_{L^{p,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} = \sup_{t>0} t\left(\frac{w(\{x \in Q : |f(x)| > t\})}{w(Q)}\right)^{\frac{1}{p}}.$$

- We have the relation $\|f\|_{L^{p,\infty}} \leq \|f\|_{L^{p,p}} = \|f\|_{L^p} \leq p\,\|f\|_{L^{p,1}}$.

The $SD_p^s(w)$ condition

- Let w be any weight in $\mathbb{R}^n$, and let $s > 0$. We say that the functional $a \in SD_p^s(w)$ for $0 \leq p < \infty$ if there is a constant C such that, for any cube Q and any family $\{Q_j\}_j$ of pairwise disjoint subcubes of Q, the following inequality holds:

$$\sum_j a(Q_j)^p w(Q_j) \leq C^p \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q).$$

The best possible constant C above is denoted by $\|a\|$.

- This conditions was introduced in [PR19]. It is very versatile and applicable to different contexts such as rectangles [CMPR22] or fractional Poincaré-Sobolev inequalities [HSMPPV22].
- The model examples of functionals satisfying this condition are $a(Q) = \ell(Q)^\alpha \left(\frac{\nu(Q)}{w(Q)}\right)^{\frac{1}{p}}$, where ν is a locally finite measure.

Theorem([PR19, LLO22])

Let $1 < p < \infty$. Consider a functional a satisfying the $SD_p^s(w)$ condition with $s > 0$ and constant $\|a\|$. Let f be a locally integrable function such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q| dx \leq a(Q)$$

for every cube Q. Then there exist a dimensional constant C_n such that, for any cube Q,

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^p w(x) dx\right)^{\frac{1}{p}} \leq C_n(1+s)\,\|a\|\,a(Q).$$

Main self-improving result

Theorem (work in preparation)

Let $1 \leq p < \infty$ and w any weight. Consider a functional $a \in SD_p^s(w)$ with $s > 0$ and constant $\|a\|$. Let f be a locally integrable function such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q| dx \leq a(Q) \tag{1}$$

and,

$$\|f - f_Q\|_{L^{p,\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq Ka(Q), \tag{2}$$

for every cube Q, where K is a uniform constant on Q. Then, for any $0 < q < \infty$,

$$\|f - f_Q\|_{L^{p,q}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq c_n \left(\frac{p}{q}\right)^{\frac{1}{q}} (K\,\|a\|)^{\frac{s}{s+1}} (1+s)^{\frac{1}{q}} a(Q), \tag{3}$$

for all cube Q.

Sketch of the proof

1. Fix $t > 1$, using (1) we can consider the local Calderón-Zygmund decomposition of the function $\frac{|f-f_Q|}{a(Q)}$ relative to the cube Q at height t. Then, there is a family of dyadic pairwise disjoint subcubes $\{Q_j\}_j$ with respect to Q.
2. As a consequence of the properties that satisfies the C-Z cubes, we prove the following inclusion, except for a set of measure zero,

$$\left\{x \in Q : \frac{|f(x) - f_Q|}{a(Q)} > c_n t\right\} \subseteq \bigcup_j \left\{x \in Q_j : \frac{|f(x) - f_{Q_j}|}{a(Q)} > t\right\}.$$

3. Taking the weight measure, using the weak Poincaré inequality (2) and the $SD_p^s(w)$ condition we obtain the key estimate,

$$w\left(\left\{x \in Q : \frac{|f(x) - f_Q|}{a(Q)} > c_n t\right\}\right)^{\frac{1}{p}} \leq K\,\|a\|\,t^{-1-\frac{1}{s}} w(Q)^{\frac{1}{p}}, \quad \text{for all } t > 1. \tag{4}$$

4. For each $L > 1$, we split the integral of the $L^{p,q}$ norm,

$$\left\|\frac{f - f_Q}{a(Q)}\right\|^q_{L^{p,q}\left(Q,\frac{w(x)dx}{w(Q)}\right)} = c_n p\int_0^L \cdots dt + c_n p\int_L^\infty \cdots dt.$$

5. The first term can be estimated trivially. For the second term we use the estimate obtained in (4). The proof concludes minimizing over $L > 1$.

Remark

The constant $\left(\frac{p}{q}\right)^{\frac{1}{q}} (K\,\|a\|)^{\frac{s}{s+1}} (1+s)^{\frac{1}{q}}$ is not sharp.

Applications to Poincaré-Sobolev inequalities

Corollary 1 (work in preparation)

Let $1 \leq p < n$, let $w \in A_r$ be a nontrivial weight ($[w]_{A_r} > 1$) with $1 \leq r \leq p$, and let p_w^* be the weighted Sobolev exponent defined by

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n\,(r + \log[w]_{A_r})}.$$

Then, there exist a dimensional constant C_n such that ,

$$\|f - f_Q\|_{L^{p_w^*,1}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq C_n [w]_{A_p}^{\frac{1}{p}} B_{w,r}\ell(Q)\,\|\nabla f\|_{L^{p,p_w^*}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \tag{5}$$

for any cube Q, and if $[w]_{A_r} \to 1^+$ the constant $B_{w,r}$ blows up in the following way

$$B_{w,r} \approx \frac{1}{([w]_{A_r} - 1)^2}.$$

Remark

The result is false for constant weights, but it is true for weights with some degeneracy, we think that this is a new phenomenon. To see that the result is false for constant weights, we can argue as follows. If the result was true for $r = 1$ and the Lebesgue measure, then that implies the global Sobolev inequality, and as a consequence we would prove the following Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*,1}(\mathbb{R}^n)$, which is false since $L^{p^*,1}(\mathbb{R}^n) \subsetneq L^{p^*,p}(\mathbb{R}^n)$ and $L^{p^*,p}(\mathbb{R}^n)$ is de smallest rearrangement invariant space that contains $W^{1,p}(\mathbb{R}^n)$.

Corollary 2 (work in preparation)

Let $0 < \delta < 1$ and let $1 \leq p < \frac{n}{\delta}$. Let $w \in A_1$ be a nontrivial weight ($[w]_{A_1} > 1$), and let $p_{\delta,w}^*$ be the weighted fractional Sobolev exponent, defined by

$$\frac{1}{p} - \frac{1}{p_{\delta,w}^*} = \frac{\delta}{n(1 + \log[w]_{A_1})}.$$

Then, there exist a dimensional constant $C_n > 0$ such that

$$\|f - f_Q\|_{L^{p_{\delta,w}^*,1}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq C_n B_w \frac{(1-\delta)^{\frac{1}{p}}}{\delta}\ell(Q)^\delta \left(\frac{1}{w(Q)}\int_Q\int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}} dy\, w(x) dx\right)^{\frac{1}{p}} \tag{6}$$

for each cube Q, and if $[w]_{A_1} \to 1^+$ the constant B_w blows up in the following way

$$B_w \approx \frac{1}{[w]_{A_r} - 1}.$$

Remark

This result is in the spirit of [HSMPPV22, Theorem 2.1], with the larger norm $L^{p_{\delta,w}^*,1}$ on the left hand side. We also obtain the interesting extra gain factor $(1-\delta)^{\frac{1}{p}}$.

References

[CMPR22] M. E. Cejas, C. Mosquera, C. Pérez, and E. Rela, *Self-improving poincaré-sobolev type functionals in product spaces*, Journal d'Analyse Mathématique (2022).

[HSMPPV22] R. Hurri-Syrjänen, J. C. Martínez-Perales, C. Pérez, and A. V. Vähäkangas, *On the BBM-phenomenon in Fractional Poincaré-Sobolev inequalities with weights*, International Mathematics Research Notices (2022).

[LLO22] A. K. Lerner, E. Lorist, and S. Ombrosi, *Operator-free sparse domination*, Forum of Mathematics. Sigma **10** (2022).

[PR19] C. Pérez and E. Rela, *Degenerate Poincaré-Sobolev inequalities*, Transactions of the American Mathematical Society **372** (2019).

Acknowledgments

This research is supported by the Basque Government through the BERC 2022-2025 program and by the Ministry of Science and Innovation: BCAM Severo Ochoa accreditation CEX2021-001142-S/ MICIN/AEI/10.13039/501100011033.