Self-improving of Poincaré-Sobolev type inequalities

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This is part of a joint work with Carlos Pérez.

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Begin with

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq c_n \ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|dx$$

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where $f_Q = \frac{1}{|Q|}\int_Q f$. This implies:

- $(1,p)$ Poincaré inequality

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq c_n \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p dx\right)^{\frac{1}{p}}$$

- (p,p) Poincaré inequality

$$\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx\right)^{\frac{1}{p}} \leq c_n \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p dx\right)^{\frac{1}{p}}$$

- Poincaré-Sobolev inequality

$$\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^{p^*} dx\right)^{\frac{1}{p^*}} \leq c_n \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p dx\right)^{\frac{1}{p}}$$

where $p^* = \frac{pn}{n-p}$ $(\frac{1}{p}-\frac{1}{p^*}=\frac{1}{n})$ for $1 \leq p < n$.

Let $w \in A_p$, for each cube Q,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq & c_n \ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|dx \\ \leq & c_n [w]_{A_p}^{\frac{1}{p}} \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}}\end{aligned}$$

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Definition

$a \in D_p(w)$ if for any cube Q and each pairwise disjoint family $\{Q_i\}_i \subset Q$,

$$\left(\sum_i a(Q_i)^p \frac{w(Q_i)}{w(Q)}\right)^{\frac{1}{p}} \leq \|a\|_{D_p} a(Q).$$

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- The model example is $a_\mu(Q) = \lambda \ell(Q)^\alpha \left(\frac{\mu(Q)}{w(Q)}\right)^{\frac{1}{p}}$ where μ is a measure and $p, \alpha > 0$.

The model example

Let $\{Q_i\}_i \subset Q$ be a disjoint family.

$$\sum_i a_\mu(Q_i)^p w(Q_i) = \lambda^p \sum_i \ell(Q_i)^{\alpha p} \frac{\mu(Q_i)}{w(Q_i)} w(Q_i)$$

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- $a_\mu \in D_p(w)$ with $\|a\|_{D_p} = 1$.

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- $a_\mu \in D_p(w)$ with $\|a\|_{D_p} = 1$.
- If $w \in A_\infty$, then $a_\mu \in D_q(w)$ for some $q > p$.

First self-improving result

Theorem (Franchi, Pérez, Wheeden 98; Canto, Pérez 21)

Let $a \in D_p(w)$ with $w \in A_\infty$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for each cube Q,

$$\|f - f_Q\|_{L^{p,\infty}\left(Q, \frac{w\,dx}{w(Q)}\right)} \leq c_n p [w]_{A_\infty} \|a\|_{D_p} a(Q).$$

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- $\|f\|_{L^{p,\infty}(X,\mu)} = \sup_{t>0} t\mu(\{x \in X : |f(x)| > t\})^{\frac{1}{p}}$.

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- $\|f\|_{L^{p,\infty}(X,\mu)} = \sup_{t>0} t\mu(\{x \in X : |f(x)| > t\})^{\frac{1}{p}}$.
- $\|f - f_Q\|_{L^{p,\infty}\left(Q, \frac{wdx}{w(Q)}\right)} \leq c_n p [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w dx\right)^{\frac{1}{p}}$.

The model example: revisited

Let $\{Q_i\}_i \subset Q$ be a disjoint family.

$$\sum_i a_\mu(Q_i)^p w(Q_i) = \lambda^p \sum_i \ell(Q_i)^{\alpha p} \mu(Q_i)$$

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$$\begin{aligned}\sum_i a_\mu(Q_i)^p w(Q_i) &= \lambda^p \sum_i \ell(Q_i)^{\alpha p} \mu(Q_i) \\ &\leq \lambda^p \sup_i \ell(Q_i)^{\alpha p} \sum_i \mu(Q_i) \\ &= \lambda^p \left(\sup_i |Q_i| \right)^{\frac{\alpha p}{n}} \sum_i \mu(Q_i)\end{aligned}$$

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&= \lambda^p \left(\sup_i |Q_i| \right)^{\frac{\alpha p}{n}} \sum_i \mu(Q_i) \\
&\leq \lambda^p \left| \cup_i Q_i \right|^{\frac{\alpha p}{n}} \frac{\mu(Q)}{w(Q)} w(Q) \\
&= \left(\frac{|\cup_i Q_i|}{|Q|} \right)^{\frac{\alpha p}{n}} a(Q)^p w(Q)
\end{aligned}$$

Definition

$a \in SD_p^s(w)$ if for any cube Q and each pairwise disjoint family $\{Q_i\}_i \subset Q$,

$$\left(\sum_i a(Q_i)^p \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p}} \leq \|a\|_{SD_p^s} \left(\frac{|\cup_i Q_i|}{|Q|} \right)^{\frac{1}{s}} a(Q).$$

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Theorem (Pérez, Rela 19)

Let $a \in SD_p^s(w)$ with $w \in A_\infty$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for each cube Q,

$$\|f - f_Q\|_{L^p\left(Q, \frac{w dx}{w(Q)}\right)} \leq c_n s \|a\|^s_{SD_p^s} a(Q).$$

Remarks

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- Recently, the conjecture has been proved true using sparse domination.

Theorem (Lerner, Lorist, Ombrosi 22)

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Main theorem

Theorem (C., Pérez)

Let $a \in SD_p^s(w)$ with $w \in A_\infty$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for each cube Q,

$$\|f - f_Q\|_{L^{p\left(1+\frac{1}{s}\right),\infty}\left(Q,\frac{wdx}{w(Q)}\right)} \leq c_n p\left(1+\frac{1}{s}\right)\|a\|_{SD_p^s}^{\frac{s}{s+1}}[w]_{A_\infty}a(Q).$$

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- We recover the theorem of [FPW98, CP21] (Idea: $SD_p^\infty = D_p$).
- We have two different proofs of the theorem.
- The result is true without the A_∞ condition.

Self-improving in Lorentz spaces

Using Kolmogorov inequality, we have for all $0 < q < \infty$,

$$\|f - f_Q\|_{L^{p,q}\left(Q, \frac{w\,dx}{w(Q)}\right)} \leq \left(\frac{p}{q}\right)^{\frac{1}{q}} (1+s)^{\frac{1}{q}} \|f - f_Q\|_{L^{p\left(1+\frac{1}{s}\right),\infty}\left(Q, \frac{w\,dx}{w(Q)}\right)}.$$

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Corollary (C., Pérez)

Let $a \in SD_p^s(w)$ with $w \in A_\infty$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for all $0 < q < \infty$ and for each cube Q,

$$\|f - f_Q\|_{L^{p,q}\left(Q, \frac{w\,dx}{w(Q)}\right)} \leq c_n p \left(1 + \frac{1}{s}\right)\left(\frac{p}{q}\right)^{\frac{1}{q}} (1+s)^{\frac{1}{q}} \|a\|^{\frac{s}{s+1}} [w]_{A_\infty} a(Q).$$

Thank you!