Weighted Poincaré-Sobolev inequalities

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This is part of a joint work with Carlos Pérez.

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1. Introduction and motivation
2. Self-improving of Poincaré-type inequalities
3. Subrepresentation formula and fractional integrals

Poincaré inequalities

Theorem ((1,1) Poincaré Inequality)

There exist a constant $c_n > 0$ such that for all cube $Q \subset \mathbb{R}^n$ and all function $f \in W^{1,1}(Q)$, the following inequality holds

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq c_n \ell(Q) \frac{1}{|Q|}\int_Q |\nabla f(x)|dx,$$

where $f_Q = \frac{1}{|Q|}\int_Q f$.

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where $f_Q = \frac{1}{|Q|}\int_Q f$.

- Subrepresentation formula: $|f(x) - f_Q| \leq c_n I_1(|\nabla f|\chi_Q)(x)$.
- $I_\alpha f(x) = \displaystyle\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$, for each $x \in \mathbb{R}^n$.

Poincaré inequalities

Proof.

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq & c_n \frac{1}{|Q|}\int_Q I_1(|\nabla f|\chi_Q)(x)dx \\ = & c_n \frac{1}{|Q|}\int_Q |\nabla f(x)| I_1(\chi_Q)(x)dx \\ \leq & c_n \ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|dx\end{aligned}$$



Poincaré inequalities

(1,1) Poincaré inequality implies:

- $(1,p)$ Poincaré inequality

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq c_n \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p dx\right)^{\frac{1}{p}}$$

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- (p,p) Poincaré inequality

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- Poincaré-Sobolev inequality

$$\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^{p^*} dx\right)^{\frac{1}{p^*}} \leq c_n \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p dx\right)^{\frac{1}{p}}$$

where $p^* = \frac{pn}{n-p}$ $(\frac{1}{p}-\frac{1}{p^*}=\frac{1}{n})$ for $1 \leq p < n$.

Motivation: Elliptic operators

$$L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... \frac{\partial^2}{\partial x_n^2} = \operatorname{div}(\nabla), \qquad Lu = 0.$$

Elliptic operators:

$$Lu = \operatorname{div}(A(x) \cdot \nabla u) = 0,$$

where $A(x)$ is a $n \times n$ matrix such that

$$c|\xi|^2 \leq A(x)\xi \cdot \xi \leq C|\xi|^2.$$

Goal: To prove local Hölder continuity of the solutions of the equation.

- Di Giorgi (1957), Nash (1957).
- Moser (1960) (Moser iteration method)
  - $(2, 2)$ Poincaré inquality.
  - Poincaré-Sobolev inequality.

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^{2^*} dx\right)^{\frac{1}{2^*}} \leq c_n \ell(Q) \left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^2 dx\right)^{\frac{1}{2}}.$$

Motivation: Degenerate elliptic operators

$$Lu = \operatorname{div}(A(x) \cdot \nabla u) = 0,$$

$$c|\xi|^2 w(x) \leq A(x)\xi \cdot \xi \leq C|\xi|^2 w(x)$$

where w is weight with some singularity.
Goal: To prove local Hölder continuity of the solutions of the equation.

Moser iteration method: Fabes, Kenig, Serapioni (1982).
Let $w \in A_2$,

- (2,2) Poincaré inequality

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^2 w dx\right)^{\frac{1}{2}} \leq c_w \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^2 w dx\right)^{\frac{1}{2}}.$$

- Poincaré-Sobolev inequality, for some $2^*_w > 2$:

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^{2^*_w} w dx\right)^{\frac{1}{2^*_w}} \leq C_w \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^2 w dx\right)^{\frac{1}{2}}.$$

Weighted Poincaré-Sobolev inequalities

Goal

To prove

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^{p_w^*}w(x)dx\right)^{\frac{1}{p_w^*}} \leq C_w \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}}$$

for an appropriate class of weights, where $p_w^* > p$.

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Goal

To prove

$$\|f-f_Q\|_{L^{p_w^*,q}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq C_w \ell(Q)\,\|\nabla f\|_{L^{p,r}\left(Q,\frac{w(x)dx}{w(Q)}\right)}$$

for an appropriate class of weights, where $p_w^* > p$, $1 \leq q, r \leq \infty$.

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for an appropriate class of weights, where $p_w^* > p$, $1 \leq q, r \leq \infty$.

Goal

Good bounds for C_w and p_w^*.

Self-improving of Poincaré-type inequalities

Let $w \in A_p$, for each cube Q,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq & c_n \ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|dx \\ \leq & c_n [w]_{A_p}^{\frac{1}{p}} \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}}\end{aligned}$$

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Definition

$a \in D_p(w)$ if for any cube Q and each pairwise disjoint family $\{Q_i\}_i \subset Q$,

$$\sum_i a(Q_i)^p w(Q_i) \leq \|a\|_{D_p}^p \, a(Q)^p w(Q).$$

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- The model example is $a_\mu(Q) = \lambda \ell(Q)^\alpha \left(\frac{\mu(Q)}{w(Q)}\right)^{\frac{1}{p}}$ where μ is a measure and $p, \alpha > 0$.

The model example

Let $\{Q_i\}_i \subset Q$ be a disjoint family.

$$\sum_i a_\mu(Q_i)^p w(Q_i) = \lambda^p \sum_i \ell(Q_i)^{\alpha p} \frac{\mu(Q_i)}{w(Q_i)} w(Q_i)$$

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Let $\{Q_i\}_i \subset Q$ be a disjoint family.

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- $a_\mu \in D_p(w)$ with $\|a\|_{D_p} = 1$.

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- $a_\mu \in D_p(w)$ with $\|a\|_{D_p} = 1$.
- If $w \in A_\infty$, then $a_\mu \in D_q(w)$ for some $q > p$.

First self-improving result

Theorem (Franchi, Pérez, Wheeden 98; Canto, Pérez 21)

Let $a \in D_p(w)$ with $w \in A_\infty$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for each cube Q,

$$\|f - f_Q\|_{L^{p,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq c_n p [w]_{A_\infty} \|a\|_{D_p} a(Q).$$

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- $\|f - f_Q\|_{L^{p,\infty}\left(Q, \frac{wdx}{w(Q)}\right)} \leq c_n p [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w dx\right)^{\frac{1}{p}}.$

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- $\|f - f_Q\|_{L^{p,\infty}\left(Q, \frac{wdx}{w(Q)}\right)} \leq c_n p [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w dx\right)^{\frac{1}{p}}.$
- Truncation method:

  $\|f - f_Q\|_{L^{p}\left(Q, \frac{wdx}{w(Q)}\right)} \leq 4c_n p [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w dx\right)^{\frac{1}{p}}.$

The model example: revisited

Let $\{Q_i\}_i \subset Q$ be a disjoint family.

$$\sum_i a_\mu(Q_i)^p w(Q_i) = \lambda^p \sum_i \ell(Q_i)^{\alpha p} \mu(Q_i)$$

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The model example: revisited

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\sum_i a_\mu(Q_i)^p w(Q_i) &= \lambda^p \sum_i \ell(Q_i)^{\alpha p} \mu(Q_i) \\
&\leq \lambda^p \sup_i \ell(Q_i)^{\alpha p} \sum_i \mu(Q_i) \\
&= \lambda^p \left(\sup_i |Q_i| \right)^{\frac{\alpha p}{n}} \sum_i \mu(Q_i) \\
&\leq \lambda^p \left| \cup_i Q_i \right|^{\frac{\alpha p}{n}} \frac{\mu(Q)}{w(Q)} w(Q) \\
&= \left(\frac{|\cup_i Q_i|}{|Q|} \right)^{\frac{\alpha p}{n}} a_\mu(Q)^p w(Q).
\end{aligned}$$

Self-improving of Poincaré-type inequalities

Definition

$a \in SD_p^s(w)$ if for any cube Q and each pairwise disjoint family $\{Q_i\}_i \subset Q$,

$$\sum_i a(Q_i)^p w(Q_i) \leq \|a\|^p_{SD_p^s} \left(\frac{|\cup_i Q_i|}{|Q|} \right)^{\frac{p}{s}} a(Q)^p w(Q).$$

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Theorem (Pérez, Rela 19)

Let $a \in SD_p^s(w)$ with $w \in A_\infty$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for each cube Q,

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^p w(x)dx\right)^{\frac{1}{p}} \leq c_n s \|a\|_{SD_p^s}^s a(Q).$$

Remarks

- Pérez and Rela conjectured that the result is true without the A_∞ condition.

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- Recently, the conjecture has been proved true using sparse domination.

Theorem (Lerner, Lorist, Ombrosi 22)

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$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for each cube Q,

$$\left(\frac{1}{w(Q)}\int_Q |f(x) - f_Q|^p w(x)dx\right)^{\frac{1}{p}} \leq c_n(1+s)\,\|a\|_{SD_p^s}\, a(Q).$$

Main theorem

Theorem (C., Pérez)

Let $a \in SD_p^s(w)$ with $w \in A_\infty$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for each cube Q,

$$\|f - f_Q\|_{L^{p\left(1+\frac{1}{s}\right),\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} \leq c_n p\left(1+\frac{1}{s}\right)\|a\|_{SD_p^s}^{\frac{s}{s+1}}[w]_{A_\infty}a(Q).$$

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Then for each cube Q,

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- We recover the theorem of [FPW98, CP21] (Idea: $SD_p^\infty = D_p$).

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- We recover the theorem of [FPW98, CP21] (Idea: $SD_p^\infty = D_p$).
- This is a Poincaré-Sobolev inequality.

Self-improving result without the A_∞ condition

Theorem (C., Pérez)

Let $a \in SD_p^s(w)$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q)$$

and,

$$\|f - f_Q\|_{L^{p,\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq K a(Q),$$

for every cube Q, where K is a uniform constant on Q. Then for each cube Q,

$$\|f - f_Q\|_{L^{p\left(1+\frac{1}{s}\right),\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq c_n \max\left\{1, K\|a\|\right\}^{\frac{s}{s+1}} a(Q).$$

- Using [LLO22] we obtain the second inequality with $K = c_n(1+s)\|a\|_{SD_p^s}$.

Proof

Fix a cube Q, and let $t > a(Q)$, then

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q) < t.$$

Consider the Calderón-Zygmund decomposition of $|f - f_Q|$ relative to the cube Q at height t. Then, there is a family of dyadic pairwise disjoint subcubes $\{Q_j\}_j$ with respect to Q, such that

- For each j,
$$t < \frac{1}{|Q_j|}\int_{Q_j} |f(x) - f_Q|dx \leq 2^n t.$$
- $$\frac{\left|\bigcup_j Q_j\right|}{|Q|} \leq \frac{a(Q)}{t}.$$
- For almost every $x \in Q \setminus \bigcup_j Q_j$,
$$|f(x) - f_Q| \leq t.$$

Proof

Let $c_n = 2^n + 1$, we have,

$$\{x \in Q : |f(x) - f_Q| > c_n t\} = \left(\bigcup_j \{x \in Q_j : |f(x) - f_Q| > c_n t\} \right)$$
$$\cup \left\{ x \in Q \setminus \bigcup_j Q_j : |f(x) - f_Q| > c_n t \right\}$$

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Let $c_n = 2^n + 1$, we have,

$$\begin{aligned}\{x \in Q : |f(x) - f_Q| > c_n t\} &= \left(\bigcup_j \{x \in Q_j : |f(x) - f_Q| > c_n t\}\right) \\ &\quad \cup \left\{x \in Q \setminus \bigcup_j Q_j : |f(x) - f_Q| > c_n t\right\} \\ &\subseteq \left(\bigcup_j \{x \in Q_j : |f(x) - f_{Q_j}| > t\}\right) \\ &\quad \cup \left\{x \in Q \setminus \bigcup_j Q_j : |f(x) - f_Q| > c_n t\right\},\end{aligned}$$

and $|\{x \in Q \setminus \cup_j Q_j : |f(x) - f_Q| > c_n t\}| = 0$.

Proof

Then,

$$w\left(\{x \in Q : |f(x) - f_Q| > c_n t\}\right) \leq \sum_j w\left(\left\{x \in Q_j : |f(x) - f_{Q_j}| > t\right\}\right)$$

Proof

Then,

$$\begin{aligned} w\left(\{x \in Q : |f(x) - f_Q| > c_n t\}\right) \leq & \sum_j w\left(\left\{x \in Q_j : |f(x) - f_{Q_j}| > t\right\}\right) \\ \leq & K^p t^{-p} \sum_j a(Q_j)^p w(Q_j) \end{aligned}$$

Proof

Then,

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\leq & K^p t^{-p} \sum_j a(Q_j)^p w(Q_j) \\
\leq & K^p t^{-p} \|a\|^p \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q)
\end{aligned}
$$

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\end{aligned}
$$

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$$\begin{aligned} w\left(\{x \in Q : |f(x) - f_Q| > c_n t\}\right) \leq & \sum_j w\left(\left\{x \in Q_j : |f(x) - f_{Q_j}| > t\right\}\right) \\ \leq & K^p t^{-p} \sum_j a(Q_j)^p w(Q_j) \\ \leq & K^p t^{-p} \|a\|^p \left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q) \\ \leq & K^p \|a\|^p \left(\frac{a(Q)}{t}\right)^{p+\frac{p}{s}} w(Q). \end{aligned}$$

This is,

$$\sup_{t > a(Q)} t \left(\frac{1}{w(Q)} w\left(\{x \in Q : |f(x) - f_Q| > t\}\right)\right)^{\frac{1}{p\left(1+\frac{1}{s}\right)}} \leq c_n \left(K \|a\|\right)^{\frac{s}{s+1}} a(Q).$$

Proof

On the other hand,

$$\sup_{t\leq a(Q)} t\left(\frac{1}{w(Q)}w\left(\{x\in Q:|f(x)-f_Q|>t\}\right)\right)^{\frac{1}{p\left(1+\frac{1}{s}\right)}}\leq a(Q).$$

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This concludes the proof,

$$\begin{aligned}\|f-f_Q\|_{L^{p\left(1+\frac{1}{s}\right),\infty}\left(Q,\frac{w(x)dx}{w(Q)}\right)} &= \sup_{t>0} t\left(\frac{1}{w(Q)}w\left(\{x\in Q:|f(x)-f_Q|>t\}\right)\right)^{\frac{1}{p\left(1+\frac{1}{s}\right)}}\\ &= \max\left\{\sup_{t\leq a(Q)}\cdots,\sup_{t>a(Q)}\cdots\right\}\\ &\leq \max\left\{c_n a(Q), c_n\left(K\|a\|\right)^{\frac{s}{s+1}}a(Q)\right\}\\ &= c_n\max\left\{1,\left(K\|a\|\right)^{\frac{s}{s+1}}\right\}a(Q).\end{aligned}$$

□

Applications: Self-improving in Lorentz spaces

Using Kolmogorov inequality, we have for all $0 < q < \infty$,

$$\|f - f_Q\|_{L^{p,q}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq \left(\frac{p}{q}\right)^{\frac{1}{q}} (1+s)^{\frac{1}{q}} \|f - f_Q\|_{L^{p\left(1+\frac{1}{s}\right),\infty}\left(Q, \frac{w(x)dx}{w(Q)}\right)}.$$

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Corollary (C., Pérez)

Let $a \in SD_p^s(w)$ with $w \in A_\infty$. Let $f \in L^1_{loc}$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q), \qquad \text{for all cube } Q.$$

Then for all $0 < q < \infty$ and for each cube Q,

$$\|f - f_Q\|_{L^{p,q}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq c_n p \left(1 + \frac{1}{s}\right)\left(\frac{p}{q}\right)^{\frac{1}{q}} (1+s)^{\frac{1}{q}} \|a\|^{\frac{s}{s+1}} [w]_{A_\infty} a(Q).$$

- In the case $q = p$, we improve the exponent of $(1+s)$ of [LLO22].

Applications to Poincaré-Sobolev inequalities

Theorem (C., Pérez)

Let $1 < p < n$, let $w \in A_r$ with $1 \leq r \leq p$ and $[w]_{A_r} > 1$, and let p_w^* be defined by

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n\left(r + \log[w]_{A_r}\right)}.$$

Then, there exists a dimensional constant C_n such that

$$\|f - f_Q\|_{L^{p_w^*,1}\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq C_n [w]_{A_p}^{\frac{1}{p}} B_{w,r} \ell(Q) \|\nabla f\|_{L^{p,p_w^*}\left(Q, \frac{w(x)dx}{w(Q)}\right)}$$

for any cube Q, where the constant $B_{w,r}$ blows up in the following way

$$B_{w,r} \approx \frac{1}{\left([w]_{A_r} - 1\right)^2}.$$

when $[w]_{A_r} \to 1^+$.

Subrepresentation formula and fractional integrals

- Subrepresentation formula: $|f(x) - f_Q| \leq c_n I_1(|\nabla f|\chi_Q)(x)$.

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- $I_\alpha f(x) = \displaystyle\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$, for each $x \in \mathbb{R}^n \qquad (0 < \alpha < n)$.
- $I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$ where $1 < p < \frac{n}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.
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Theorem (Pérez, Rela 2023)

Let $1 \leq p < n$ and let $w \in A_{p,p^*}$. Then there exist a constant $c_{n,p} > 0$ such that for any cube Q,

$$\left(\int_Q (|f(x) - f_Q|w(x))^{p^*}\, dx\right)^{\frac{1}{p^*}} \leq c_{n,p}[w]_{A_{p,p^*}}^{\frac{1}{n'}} \left(\int_Q (|\nabla f(x)|w(x))^p\, dx\right)^{\frac{1}{p}}.$$

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- These are inequalities of a different nature!

A_p weights

Theorem (Precise open property, Hytönen, Pérez, Rela 2012)

Let $1 < p < \infty$ and let $w \in A_p$. We define the quantity

$$\varepsilon = \frac{p-1}{\tau_n [\sigma]_{A_\infty}},$$

where $\tau_n = 2^{n+1}$ and $\sigma = w^{1-p'}$. Then, $w \in A_{p-\varepsilon}$ and $[w]_{A_{p-\varepsilon}} \leq 2^{p-1}[w]_{A_p}$.

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- Remark: If $w \in A_\infty \setminus A_1$, then $w \notin A_{\ell_w}$.

Fractional integrals and A_p weights

Motivated by a work of Alberico, Cianchi and Sbordone (2009).

Theorem (C.)

Let $n \geq 2$, $\alpha \in (0, n)$ and $1 < p < \frac{n}{\alpha}$. Let $w \in A_p$ and consider the exponent q_r defined by the relation

$$\frac{1}{p} - \frac{1}{q_r} = \frac{\alpha}{nr}$$

where $r \in I(w)$. Then, there exist a positive constant $C = C(\alpha, p, n)$ such that

$$\left(\frac{1}{w(B)} \int_B |I_\alpha f|^{q_r} w dx \right)^{\frac{1}{q_r}} \leq C [w]_{A_r}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{q_r}} r(B)^\alpha \left(\frac{1}{w(B)} \int_B |f|^p w dx \right)^{\frac{1}{p}},$$

for any ball $B \subset \mathbb{R}^n$ of radius $r(B)$ and every function $f \in L^p(B, w)$.

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Fractional integrals and A_p weights

Sketch of the proof.

- Weighted Hedberg's inequality: Let $1 < p < \frac{n}{\alpha}$ and let w be a weight. Let $1 < r \leq p$ and consider q defined by

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Then, there exist a constant $C = C(n, \alpha, r, p) > 0$ such that

$$|I_\alpha f(x)| \leq C M f(x)^{\frac{p}{q}} \|f\|_{L^p(B,w)}^{1-\frac{p}{q}} \left(\int_B w(y)^{1-r'} dy \right)^{\frac{\alpha}{nr'}}$$

for any ball B such that $0 < \int_B w(y)^{1-r'} dy < \infty$, and every function $f \in L^p(B, w)$.

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- Weighted norm inequalities for the Hardy-Littlewood maximal operator.



Applications to Poincaré-Sobolev inequalities

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- The case $r = 1 \in I(w)$ (this means $w \in A_1$) of the theorem solves a conjecture of [PR19] about the sharp exponent in the A_1 constant.

¡Muchas gracias!