Self-improving properties of Poincaré-type inequalities

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This is part of a joint work with Carlos Pérez (BCAM, UPV/EHU) and Linfei Zheng (Tianjin University).

EARCO 2026

Plan for the talk

1. Classical and weighted Poincaré-Sobolev inequalities.
2. Generalized Poincaré inequalities and self-improvement.
3. The new A_r self-improving theorem.
4. Applications and extensions.

Introduction and motivation

Poincaré inequalities

Theorem ($(1,1)$ Poincaré inequality)

There exists a dimensional constant $c_n > 0$ such that, for every cube $Q \subset \mathbb{R}^n$ and every $f \in W^{1,1}(Q)$,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\, dx \leq c_n \ell(Q)\, \frac{1}{|Q|}\int_Q |\nabla f(x)|\, dx,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$.

- Subrepresentation formula:

$$|f(x) - f_Q| \leq c_n I_1(|\nabla f|\chi_Q)(x), \qquad x \in Q.$$

where $I_\alpha g(x) = \displaystyle\int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}}\, dy.$

Poincaré inequalities

The $(1,1)$ Poincaré inequality implies:

- $(1,p)$ Poincaré inequality: for $1 \leq p < \infty$,

$$\frac{1}{|Q|}\int_Q |f - f_Q|\, dx \lesssim \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p\, dx\right)^{\frac{1}{p}}.$$

- (p,p) Poincaré inequality: for $1 \leq p < \infty$,

$$\left(\frac{1}{|Q|}\int_Q |f - f_Q|^p\, dx\right)^{\frac{1}{p}} \lesssim \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p\, dx\right)^{\frac{1}{p}}.$$

- Poincaré-Sobolev inequality: for $1 \leq p < n$,

$$\left(\frac{1}{|Q|}\int_Q |f - f_Q|^{p^*}\, dx\right)^{\frac{1}{p^*}} \lesssim \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p\, dx\right)^{\frac{1}{p}}, \qquad \frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}.$$

Motivation: Elliptic operators

$$L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots \frac{\partial^2}{\partial x_n^2} = \operatorname{div}(\nabla), \qquad Lu = 0.$$

Elliptic operators:

$$Lu = \operatorname{div}(A(x) \cdot \nabla u) = 0,$$

where $A(x)$ is a $n \times n$ matrix such that

$$c|\xi|^2 \leq A(x)\xi \cdot \xi \leq C|\xi|^2.$$

Goal: To prove local Hölder continuity of the solutions of the equation.

- Di Giorgi (1957), Nash (1957).
- Moser (1960) (Moser iteration method)
  - $(2, 2)$ Poincaré inquality.
  - Poincaré-Sobolev inequality.

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^{2^*} dx\right)^{\frac{1}{2^*}} \leq c_n \ell(Q) \left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^2 dx\right)^{\frac{1}{2}}.$$

Motivation: Degenerate elliptic operators

$$Lu = \operatorname{div}(A(x) \cdot \nabla u) = 0,$$

$$c|\xi|^2 w(x) \leq A(x)\xi \cdot \xi \leq C|\xi|^2 w(x)$$

where w is a weight.

Goal: To prove local Hölder continuity of the solutions of the equation.

Moser iteration method: Fabes, Kenig, Serapioni (1982).

Let $w \in A_2$,

- (2,2) Poincaré inequality

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^2 w dx\right)^{\frac{1}{2}} \leq c_w \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^2 w dx\right)^{\frac{1}{2}}.$$

- Poincaré-Sobolev inequality, for some $2_w^* > 2$:

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^{2_w^*} w dx\right)^{\frac{1}{2_w^*}} \leq C_w \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^2 w dx\right)^{\frac{1}{2}}.$$

Generalized Poincaré inequalities

Self-improving of Poincaré-type inequalities

Generalized Poincaré inequality

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le a(Q), \qquad Q\in\mathcal{Q}.$$

Here $\mathcal{Q}$ denotes the family of all cubes and

$$a:\mathcal{Q}\longrightarrow(0,\infty)$$

is a functional on cubes. The problem becomes:

Self-improvement problem

Which geometric conditions on a imply a self-improvement of the LHS with respect to a measure $w(x)\,dx$?

This point of view goes back to Franchi, Pérez and Wheeden 1998.

The model example

Let $w \in A_p$. From the $(1,1)$ Poincaré inequality and the A_p condition,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx &\le c_n \ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|\,dx \\ &\le c_n [w]_{A_p}^{\frac{1}{p}} \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\,dx\right)^{\frac{1}{p}} \\ &:= a_f(Q).\end{aligned}$$

More generally, we keep in mind functionals of the form

$$a(Q) = \ell(Q)^\alpha \left(\frac{\mu(Q)}{w(Q)}\right)^{\frac{1}{p}},$$

where μ is a locally finite measure and $\alpha, p > 0$. The previous example corresponds to $\alpha = 1$ and $d\mu = |\nabla f|^p w\,dx$.

Self-improving of Poincaré-type inequalities

Definition

We say that $a \in D_p(w)$ if, for every cube Q and every pairwise disjoint family $\{Q_j\}_j \subset Q$,

$$\sum_j a(Q_j)^p w(Q_j) \leq C^p \, a(Q)^p w(Q).$$

- This condition was introduced by Franchi, Pérez and Wheeden in the setting of spaces of homogeneous type.
- The best constant is denoted by $\|a\|_{D_p(w)}$, and $\|a\|_{D_p(w)} \geq 1$.
- For $a(Q) = \ell(Q)^\alpha \left(\frac{\mu(Q)}{w(Q)}\right)^{\frac{1}{p}}$,

$$\sum_j a(Q_j)^p w(Q_j) = \sum_j \ell(Q_j)^{\alpha p} \mu(Q_j) \leq \ell(Q)^{\alpha p} \mu(Q) = a(Q)^p w(Q).$$

Hence $a \in D_p(w)$ with constant $\|a\|_{D_p(w)} = 1$.

Self-improving of Poincaré-type inequalities with $D_p(w)$

Franchi, Pérez and Wheeden proved this self-improvement in the general setting of spaces of homogeneous type, with exponential dependence on the A_∞ constant.

In the Euclidean setting with cubes, Canto and Pérez later improved this to linear dependence on $[w]_{A_\infty}$.

Theorem (Franchi, Pérez, Wheeden 98; Canto, Pérez 21)

Let $w \in A_\infty$ and let $a \in D_p(w)$. Let $f \in L^1_{\mathrm{loc}}$ be such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \leq a(Q)$$

for every cube Q. Then, for every cube Q,

$$\|f - f_Q\|_{L^{p,\infty}(Q,\frac{w\,dx}{w(Q)})} \lesssim_n p[w]_{A_\infty}\,\|a\|_{D_p(w)}\,a(Q).$$

Self-improving of Poincaré-type inequalities

Pérez and Rela introduced a stronger condition that preserves smallness.

Definition

We say that $a \in SD_p^s(w)$ if, for every pairwise disjoint family $\{Q_j\}_j \subset Q$,

$$\sum_j a(Q_j)^p w(Q_j) \leq \|a\|^p_{SD_p^s(w)} \left(\frac{|\bigcup_j Q_j|}{|Q|}\right)^{\frac{p}{s}} a(Q)^p w(Q).$$

The same model example functional satisfies this stronger property. For every pairwise disjoint family $\{Q_j\}_j \subset Q$,

$$\sum_j a(Q_j)^p w(Q_j) = \sum_j \ell(Q_j)^{\alpha p} \mu(Q_j) \leq \left(\sup_j |Q_j|\right)^{\frac{\alpha p}{n}} \sum_j \mu(Q_j)$$

$$\leq |\cup_j Q_j|^{\frac{\alpha p}{n}} \mu(Q) = \left(\frac{|\cup_j Q_j|}{|Q|}\right)^{\frac{p}{n/\alpha}} a(Q)^p w(Q).$$

Hence $a \in SD_p^{n/\alpha}(w)$ with constant $\|a\|_{SD_p^{n/\alpha}(w)} = 1$.

Self-improving of Poincaré-type inequalities with $SD_p^s(w)$

Pérez and Rela proved the following self-improving result assuming $w \in A_\infty$.

Later, Lerner, Lorist and Ombrosi showed that the same conclusion is valid for arbitrary weights.

Theorem (Pérez, Rela 19; Lerner, Lorist, Ombrosi 22)

Let $a \in SD_p^s(w)$ and let $f \in L^1_{\mathrm{loc}}$ be such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \leq a(Q)$$

for every cube Q. Then, for every cube Q,

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^p w\,dx\right)^{\frac{1}{p}} \lesssim_n (1+s)\,\|a\|_{SD_p^s(w)}\,a(Q).$$

A natural question

For the $SD_p^s(w)$ condition, the A_∞ assumption can be removed.
One may ask whether the same thing happens for the weaker $D_p(w)$ condition.

Question

Does the $D_p(w)$ self-improving theorem remain true for arbitrary weights?

The answer is negative. The key example is the constant functional

$$a(Q) = \|f\|_{\mathrm{BMO}} \in D_p(w) \qquad \text{for every } p \text{ and every weight } w$$

If the $D_p(w)$–theorem were valid for arbitrary weights, we would obtain

$$\|f\|_{\mathrm{BMO}_{w,w}} \lesssim \|f\|_{\mathrm{BMO}}\,.$$

But, by the characterization of Ombrosi, Pérez, Rela and Rivera-Ríos (2020), the embedding

$$\mathrm{BMO} \hookrightarrow \mathrm{BMO}_{w,w}$$

holds if and only if $w \in A_\infty$. Moreover, we show that the linear dependence $[w]_{A_\infty}$ is sharp.

From Generalized Poincaré to Poincaré–Sobolev Inequalities

The $SD_p^s(w)$ theorem is valid for arbitrary weights:

$$a \in SD_p^s(w) \quad \Longrightarrow \quad L^p(w)\text{-self-improvement.}$$

In this work, we show that if the weight satisfies an additional condition such as $w \in A_r$, then one can go beyond $L^p(w)$ and obtain a generalized Poincaré–Sobolev improvement.

Main result

Main theorem

Theorem (C., Pérez, Zheng)

Let $1 \leq p, r, s < \infty$, let $w \in A_r$, and assume

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq a(Q)$$

for every cube Q. If $a \in SD_p^s(w)$ and $p < rs$, then for every cube Q,

$$\|f - f_Q\|_{L^{p^*_{r,s},\infty}(Q, \frac{w\,dx}{w(Q)})} \leq c_n p^*_{r,s} [w]_{A_\infty} [w]_{A_r}^{\frac{1}{rs}} \|a\| \; a(Q),$$

where

$$\frac{1}{p} - \frac{1}{p^*_{r,s}} = \frac{1}{rs}.$$

- This is a generalized Poincaré-Sobolev inequality. The parameter s plays the role of the dimension.
- The proof is based on a good-lambda argument.

Remarks

For the model functional

$$a_f(Q) = c_n [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx \right)^{\frac{1}{p}} \in SD_p^n(w).$$

Thus the parameter s in the abstract theorem behaves as a dimension. Therefore the previous result gives

$$\|f - f_Q\|_{L^{p^*_{r,n},\infty}\left(Q, \frac{w\,dx}{w(Q)}\right)} \lesssim \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx \right)^{\frac{1}{p}},$$

where the exponent is

$$\frac{1}{p} - \frac{1}{p^*_{r,n}} = \frac{1}{nr}.$$

This exponent is sharp for the whole class A_r. Indeed, in the range $1 \leq r \leq p$, we can prove that the corresponding estimate cannot hold for all $w \in A_r$ with $L^{q,\infty}$ on the left-hand side for any $q > p^*_{r,n}$.

Main theorem

In the critical and supercritical range $p \geq rs$, we obtain exponential integrability.

Theorem (C., Pérez, Zheng)

Let $1 \leq p, r, s < \infty$, let $w \in A_r$, and assume

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \leq a(Q)$$

for every cube Q. If $a \in SD_p^s(w)$ and $p \geq rs$, then for every cube Q,

$$\|f - f_Q\|_{\exp L(Q, \frac{w\,dx}{w(Q)})} \leq c_{n,p} [w]_{A_\infty} [w]_{A_r}^{\frac{1}{p}} \|a\| \, a(Q).$$

Idea of the proof: For every $q > p$, by using the previous case we can prove

$$\|f - f_Q\|_{L^{q,\infty}(Q, \frac{w\,dx}{w(Q)})} \lesssim_n q \, [w]_{A_\infty} [w]_{A_r}^{\frac{1}{p}} \|a\| \, a(Q).$$

If $\|g\|_{L^{q,\infty}} \leq \gamma q$ for every $q > p_0$, then $g \in \exp L$ and $\|g\|_{\exp L} \lesssim p_0 \gamma$.

Applications

Application 1: weighted Poincaré-Sobolev inequalities

Starting from

$$\frac{1}{|Q|}\int_Q |f-f_Q|\,dx \le c_n [w]_{A_p}^{\frac{1}{p}}\,\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\,dx\right)^{\frac{1}{p}},$$

we apply the theorem with

$$a(Q)=\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\,dx\right)^{\frac{1}{p}} \in SD_p^n(w).$$

If $w\in A_r$, $1\le r\le p<nr$, and $\frac{1}{p}-\frac{1}{p_r^*}=\frac{1}{nr}$, then

$$\|f-f_Q\|_{L^{p_r^*,\infty}(Q,\frac{w\,dx}{w(Q)})} \le c_n p_r^* [w]_{A_\infty}[w]_{A_p}^{\frac{1}{p}}[w]_{A_r}^{\frac{1}{rn}} \cdot \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\,dx\right)^{\frac{1}{p}}.$$

Application 1: weighted Poincaré-Sobolev inequalities

The truncation method upgrades the weak estimate to the Lorentz norm:

$$\|f - f_Q\|_{L^{p_r^*,p}(Q,\frac{w\,dx}{w(Q)})} \leq c_n p_r^* [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{rn}} \cdot \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w\,dx \right)^{\frac{1}{p}}.$$

- The second Lorentz index p is optimal.
- In the unweighted case this is sharp even among rearrangement-invariant target spaces.
- Critical/supercritical case $p \geq nr$: exponential endpoint

$$\|f - f_Q\|_{\exp L(Q,\frac{w\,dx}{w(Q)})} \lesssim [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w\,dx \right)^{\frac{1}{p}}.$$

- The same works for pairs of weights $(u, v) \in A_p$.

Application 2: fractional Poincaré-Sobolev inequalities

We start from the weighted BBM-type Poincaré inequality of Myyryläinen–Pérez–Weigt (2024): for $w \in A_p$ and $0 < \delta < 1$,

$$\frac{1}{|Q|}\int_Q |f - f_Q|\,dx \le a(Q),$$

where

$$a(Q) = c_n [w]_{A_p}^{\frac{1}{p}} (1-\delta)^{\frac{1}{p}} \ell(Q)^{\delta} \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,w(x)\,dx \right)^{\frac{1}{p}}.$$

We can prove that this functional satisfies $a \in SD_p^{n/\delta}(w)$. By the self-improving result, we obtain

$$\|f - f_Q\|_{L^{p^*_{\delta,r},\infty}(Q,\frac{w\,dx}{w(Q)})} \lesssim p^*_{\delta,r} [w]_{A_\infty} [w]_{A_r}^{\frac{\delta}{nr}} a(Q),$$

where

$$\frac{1}{p} - \frac{1}{p^*_{\delta,r}} = \frac{\delta}{nr}.$$

Application 2: fractional Poincaré-Sobolev inequalities

Using the truncation method (Dyda, Lehrbäck, Vähäkangas 2023), we can improve the left-hand side to $L^{p^*_{\delta,r},p}$ and we obtain

$$\|f - f_Q\|_{L^{p^*_{\delta,r},p}\left(Q,\frac{w\,dx}{w(Q)}\right)} \leq c_n p^*_{\delta,r} [w]_{A_\infty} [w]_{A_p}^{\frac{1}{p}} [w]_{A_r}^{\frac{\delta}{nr}} (1-\delta)^{\frac{1}{p}}$$
$$\cdot \ell(Q)^\delta \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, w(x)\, dx \right)^{\frac{1}{p}}.$$

- We obtain a weighted fractional Poincaré–Sobolev inequality while preserving the BBM factor $(1-\delta)^{\frac{1}{p}}$.
- In the unweighted case, the Lorentz refinement $L^{p^*_\delta,p}$ is optimal among rearrangement-invariant spaces.
- The same works for pairs of weights $(u,v) \in A_p$.

Application 3: Lorentz norm of the gradient

Assume that $w \in A_r$ with $1 \leq r \leq p$. Then $w \in A_r \subset A_p = A_{p,p_r^*}$ where A_{p,p_r^*} denotes the Chung–Hunt–Kurtz class adapted to Lorentz norms.
Thus we can start with a smaller norm of the gradient

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f - f_Q|\,dx &\leq c_n \ell(Q) \frac{1}{|Q|}\int_Q |\nabla f(x)|\,dx \\ &\leq c_n [w]_{A_{p,p_r^*}}^{\frac{1}{p}} \ell(Q) \|\nabla f\|_{L^{p,p_r^*}\left(Q,\frac{w\,dx}{w(Q)}\right)}.\end{aligned}$$

Applying the self-improving result, we obtain

$$\begin{aligned}\|f - f_Q\|_{L^{p_r^*,\infty}\left(Q,\frac{w\,dx}{w(Q)}\right)} \leq c_n p_r^* [w]_{A_\infty} [w]_{A_{p,p_r^*}}^{\frac{1}{p}} [w]_{A_r}^{\frac{1}{rn}} \\ \cdot \ell(Q) \|\nabla f\|_{L^{p,p_r^*}\left(Q,\frac{w\,dx}{w(Q)}\right)}.\end{aligned}$$

- The truncation method is not available

Further extensions

- **Higher-order oscillations:** replace f_Q by $P_Q^m f$, where $P_Q^m f$ is the projection of f onto the space of polynomials of degree at most m. This leads to Poincaré–Sobolev inequalities involving higher-order derivatives.
- **Vector-valued functions:** $f : \mathbb{R}^n \to \ell_q$, oscillation measured in the ℓ_q-norm.
- **Rectangles:** cubes replaced by rectangles with axis-parallel sides; new $SD^s_{p,\mathcal{R}}(w)$ condition adapted to rectangles.

Muchas gracias!