# Weighted Poincaré-Sobolev inequalities via fractional integration

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July 9, 2024

# Poincaré-Sobolev inequalities

## Theorem (Poincaré-Sobolev Inequality)

*Let $n \geq 2$ and $1 \leq p < n$. There exist a constant $c_{n,p} > 0$ such that for all ball $B \subset \mathbb{R}^n$ and all function $f \in W^{1,p}(B)$, the following inequality holds*

$$\left( \frac{1}{|B|} \int_B |f(x) - f_B|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq c_{n,p} r(B) \left( \frac{1}{|B|} \int_B |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}},$$

*where $f_B = \frac{1}{|B|} \int_B f$ and $p^* = \frac{np}{n-p}$ $(\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n})$.*

## Definition (Riesz potential)

The Riesz potential of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

for each $x \in \mathbb{R}^n$, where $\alpha \in (0, n)$.

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- $I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$ for $1 < p < \frac{n}{\alpha}$ where $q = \frac{np}{n-\alpha p}$ $\left(\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}\right)$.
- $I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^{q,\infty}(\mathbb{R}^n)$ for $1 \leq p < \frac{n}{\alpha}$.

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- $I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^{q,\infty}(\mathbb{R}^n)$ for $1 \leq p < \frac{n}{\alpha}$.

## Theorem (Subrepresentation formula)

*There exists $c_n > 0$ such that for each ball $B$,*

$$|f(x) - f_B| \leq c_n I_1(|\nabla f|\chi_B)(x), \qquad x \in B.$$

# Poincaré-Sobolev inequalities

## Proof.

Let $1 \leq p < n$,

$$\begin{aligned}\left\| f - f_B \right\|_{L^{p^*,\infty}(B,\frac{dx}{|B|})} &\leq c_n \left\| I_1(|\nabla f|\chi_B) \right\|_{L^{p^*,\infty}(B,\frac{dx}{|B|})} \\ &= c_n \frac{r(B)}{|B|^{\frac{1}{p}}} \left\| I_1(|\nabla f|\chi_B) \right\|_{L^{p^*,\infty}(B,dx)} \\ &\leq c_{n,p} r(B) \left( \frac{1}{|B|} \int_B |\nabla f(x)|^p dx \right)^{\frac{1}{p}}.\end{aligned}$$

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Using the truncation method, we obtain

$$\left( \frac{1}{|B|} \int_B |f(x) - f_B|^{p^*} dx \right)^{\frac{1}{p^*}} \leq C_{n,p} r(B) \left( \frac{1}{|B|} \int_B |\nabla f(x)|^p dx \right)^{\frac{1}{p}}.$$

□

# Motivation: Degenerate elliptic operators

$$Lu = \operatorname{div}(A(x) \cdot \nabla u), \qquad Lu = 0$$

$$c|\xi|^2 w(x) \leq A(x)\xi \cdot \xi \leq C|\xi|^2 w(x)$$

where $w$ is a weight with some singularity.
**Goal:** To prove local Hölder continuity of the solutions of the equation.

Moser iteration method: Fabes, Kenig, Serapioni (1982).
Let $w \in A_2$,

- Weighted Poincaré-Sobolev inequality, for some $2_w^* > 2$:

$$\left(\frac{1}{w(B)}\int_B |f - f_B|^{2_w^*} w dx\right)^{\frac{1}{2_w^*}} \leq C_w r(B) \left(\frac{1}{w(B)}\int_B |\nabla f|^2 w dx\right)^{\frac{1}{2}}.$$

# Weighted Poincaré-Sobolev inequalities

## Goal

To prove

$$\left(\frac{1}{w(B)}\int_B |f(x)-f_B|^{p_w^*} w(x)dx\right)^{\frac{1}{p_w^*}} \leq C_w r(B)\left(\frac{1}{w(B)}\int_B |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}}$$

for an appropriate class of weights, where $p_w^* > p$.

## Goal

Good bounds for $C_w$ and $p_w^*$.

# Fractional integrals and $A_p$ weights

Motivated by the work of Alberico, Cianchi, and Sbordone (2009).

## Theorem (C.)

*Let $n \geq 1$, $\alpha \in (0,n)$ and $1 < p < \frac{n}{\alpha}$. Let $w \in A_r$ with $1 \leq r \leq p$. Consider $q_r$ defined by the relation*

$$\frac{1}{p} - \frac{1}{q_r} = \frac{\alpha}{n} \frac{\tau_n [\sigma]_{A_\infty}}{1 + r(\tau_n [\sigma]_{A_\infty} - 1)},$$

*where $\sigma = w^{1-r'}$ and $\tau_n = 2^{n+1}$ if $r > 1$ and $q_1 = p_\alpha^* = \frac{np}{n - \alpha p}$. Then, there exists a constant $C > 0$ such that*

$$\left( \frac{1}{w(B)} \int_B |I_\alpha f(x)|^{q_r} w(x) dx \right)^{\frac{1}{q_r}} \leq C \frac{p_\alpha^*}{\alpha} [w]_{A_r}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{q_r}} r(B)^\alpha \left( \frac{1}{w(B)} \int_B |f(x)|^p w(x) dx \right)^{\frac{1}{p}},$$

*for any ball $B \subset \mathbb{R}^n$ and every function $f \in L^p(B, w)$.*

# Fractional integrals and $A_p$ weights

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*where $\sigma = w^{1-r'}$ if $r > 1$ and $q_1 = p_\alpha^* = \frac{np}{n - \alpha p}$. Then, there exists a dimensional constant $c_n > 0$ such that*

$$\left\| I_\alpha f \right\|_{L^{q_r,\infty}\left(B, \frac{w(x)dx}{w(B)}\right)} \leq \frac{c_n}{\alpha} p_\alpha^* [w]_{A_r}^{\frac{1}{p}} r(B)^\alpha \left( \frac{1}{w(B)} \int_B |f(x)|^p w(x) dx \right)^{\frac{1}{p}},$$

*for any ball $B \subset \mathbb{R}^n$ and every function $f \in L^p(B,w)$.*

# Weighted Hedberg's inequality

## Lemma (C.)

*Let $n \geq 1$, $\alpha \in (0, n)$, let $1 < p < \frac{n}{\alpha}$ and let $w$ be a weight. Let $1 < r \leq p$ and consider $q$ defined by the relation*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} \frac{1}{r}.$$

*Then, there exists a dimensional constant $c_n > 0$ such that for any ball $B$ such that $0 < \int_B w(y)^{1-r'} dy < \infty$ and every function $f \in L^p(B, w)$, we have*

$$|I_\alpha f(x)| \leq \frac{c_n}{\alpha} p_\alpha^* M f(x)^{\frac{p}{q}} \|f\|_{L^p(B,w)}^{1-\frac{p}{q}} \left( \int_B w(y)^{1-r'} dy \right)^{\frac{\alpha}{nr'}},$$

*for any $x \in B$.*

# Applications to Poincaré-Sobolev inequalities

## Theorem (C.)

*Let $n \geq 2$ and let $1 \leq p < n$. Let $w \in A_r$ for some $1 \leq r \leq p$ and consider $q_r$ defined by the relation*

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*for any ball $B$, and the exponent $\frac{1}{p}$ on the $A_r$ constant is sharp.*

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- The case $r = 1$ ($w \in A_1$) of the theorem answers positively to a conjecture of Pérez-Rela (2019) about the sharp exponent in the $A_1$ constant.

# Weighted Sobolev exponent

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where $\ell_w := \inf I(w)$.

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## Definition

Let $n \geq 2$, $1 \leq p < n$ and let $w \in A_\infty$. We define the weighted Sobolev exponent $p_w^*$ by the relation

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n} \frac{1}{\ell_w}.$$

# Weighted Sobolev exponent

## Corollary (C.)

*Let $1 \leq p < n\ell_w$ and let $w \in A_p$. Then, for each $1 \leq q < p^*_w$, we have*

$$\left(\frac{1}{w(B)}\int_B |f(x)-f_B|^q w(x)dx\right)^{\frac{1}{q}} \leq C_w r(B)\left(\frac{1}{w(B)}\int_B |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{p}}$$

*for any ball $B$, where $C_w$ is of the form $c_n p^* [w]_{A_r}^{\frac{1}{p}}$ for some $r = r(q) > \ell_w$.*

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*We can find a weight $w \in A_p$ such that if the inequality*

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*holds for all Lipschitz function $f$, then $q \leq p_w^*$.*

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*holds for all Lipschitz function $f$, then $q \leq p_w^*$.*

- We do **not** know if the inequality is true at the endpoint $q = p_w^*$.

Thank you!